Optimal Cash Management with Payables Finance

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Payables finance, also known as reverse factoring or supply chain finance, is a form of trade finance arrangement that provides a supplier with the option to receive a buyer's payables early while allowing the buyer to extend its payment due date. The recent adoption of the blockchain technology has the potential to make payables finance more efficient and secure. In this paper, we study the supplier's optimal cash policy under such a "frictionless" payables finance arrangement. Our work extends the classic cash flow management literature in two fronts: 1) we introduce the salient features of payables finance into the cash flow problem; and 2) we consider a more realistic *integrated cash balance* model, that is, all interest gains and costs are allowed to accrue together with the cash balance in a single sum. We find the optimal cash policy possesses the "non-borrow-up-to" and "non-invest-down-to" features that differ from the classic (L, U) policy known in the literature. We further quantify the value of payables finance to the supplier and determine the equilibrium payment term extension for the buyer. We show that it is the cash liquidity enabled by payables finance to hedge against cash flow uncertainty that generates value to the supplier. To tackle the computational challenge of the problem, we derive easy-to-compute heuristic policies and system bounds. Numerical studies show that heuristic policies achieve near-optimal performance. Finally, we present results from applying our model to a data set obtained from a major US chemicals company.

Key words: payables finance, supply chain finance, reverse factoring, cash management, blockchain

1. Introduction

Payables finance, also known as reverse factoring or supply chain finance, has seen a wide adoption by companies over the past decade.[1] A recent McKinsey report projected a broader adoption of payables finance in industry, with an estimated \$2 trillion in readily financeable payables in supply chains (Herath 2015). According to the description by Global Supply Chain Finance Forum (2020), "[payables finance] provides a seller of goods or services with the option of receiving the discounted value of receivables (represented by outstanding invoices) [from a third-party financial

[1] We adopt the term "payables finance" by following the name convention set by Global Supply Chain Finance Forum (2020)—"this technique is subject to a number of naming conventions, as is clear from the number of synonyms recorded; the Forum decided that the term Payables Finance is a generic and neutral expression that captures the essence of the technique."

intermediary] prior to their actual due date and typically at a financing cost aligned with the credit risk of the buyer. The payable continues to be due by the buyer until its due date." The arrangement is settled on the due date when the financed portion is repaid by the buyer to the financial intermediary (such as a bank or a FinTech platform).

According to Bonzani et al. (2018), with payables finance, small suppliers could lower their cost of financing by 30% on average. Such supplier benefit also allows buyers to negotiate extension of payment terms, further freeing up working capital for buyers. For example, it was estimated that Proctor & Gamble could free up as much as $2 billion in cash by extending its payment terms from 45 days to 75 days under payables finance (Ng 2013). In 2019, Keurig Dr Pepper, a major US coffee and soda manufacturer, was able to extend its payment terms up to 360 days and defer payments worth $2.1 billion under payables finance (Eaglesham 2020). In addition, third-party financial intermediaries can earn extra interest revenue by facilitating payables finance arrangements between suppliers and buyers. As such, payables finance enables a "win-win-win solution" for all three parties involved in the arrangement (Esty et al. 2017).

There is a recent wave of adoption of the blockchain technology in payables finance implementation, where a distributed, decentralized public ledger is used to record transactions that are immutable and can be transparent to all parties involved. For example, Ledger Insights (2018a) reported that Ant Duo-Chain, a subsidiary of Ant Group (who owns China's largest digital payment platform Alipay), applied the blockchain technology to payables finance and was able to help small suppliers receive payment within a second, which dramatically shortened the traditional three month payment period for such suppliers. It also estimated that traditional payables finance service could only finance about 15% of suppliers, but with blockchain technology, it could expand the service to about 85% of suppliers. Tencent, Ant Group's main rival in China, also announced the launch of WeChain, a blockchain-based payables finance platform for small and medium-sized enterprises (Buck 2019). The promise of the blockchain technology appears to make payables finance more efficient and secure, enabling frictionless transactions among all parties involved. In this paper, we focus on such a frictionless payables finance setting enabled by the blockchain technology.

Cash flow management is often regarded as the foundation of financial stability (Thangavelu 2021, Asif 2021). Poor cash flow management can lead to insufficient cash reserves, excess debit, and increased risk of putting small companies out of business. The COVID-19 pandemic has made cash flow management even more critical. Many firms faced heightened cash flow uncertainty due to large cash flow shock, arisen from random loss from operations and/or unpaid/late scheduled invoice payments during the pandemic (Shen 2020). According to Corporate Cash Management

Playbook (2021), 54% of CFOs considered cash flow management a top challenge, especially for the small suppliers considered in this paper.

Despite the growing importance of payables finance for small suppliers, as well as the potential of its blockchain implementation in reaching greater number of suppliers, our understanding of the supplier's cash flow management under a frictionless payables finance arrangement is still quite limited. In this paper, we focus on studying the following two research questions: First, what is the supplier's optimal cash management policy with payables finance? Second, how can we quantify the value of payables finance to the supplier and also determine the equilibrium payment term extension for the buyer in the frictionless blockchain setting?

Specifically, we consider a supplier who faces a random cash flow before the payment due date of the payables finance arrangement. In each period, the supplier can either raise cash level by drawing from payables finance and/or borrowing additional short-term loans, or lower cash level by investing to earn risk-free interest. The adjusted cash level is used to meet the random cash outflow in the period, subject to potential cost penalty if the ending balance is negative. The supplier's objective is to maximize its discounted total cash balance at the end of the payment due date. We show that this problem can be transformed into a cash flow cost minimization problem with a dynamic program formulation.

In the classic random cash flow literature (e.g., Eppen and Fama 1969, Neave 1970, Chen and Simchi-Levi 2009), a simplifying assumption was adopted such that the interest gains and costs are accrued and evaluated in a separate account from the cash balance itself. Such a decoupling assumption makes the cash flow problem more tractable in certain model settings. However, we find that it does not make the optimal policy in our payables finance problem more amenable for computation, suggesting that the problem complexity stems from the payables finance arrangement itself.

To reveal new insights about the cash flow management problem, we take a step further to relax the decoupling assumption by considering a more realistic *integrated cash balance* model, that is, all interest gains and costs are allowed to accrue together with the cash balance in a single sum. We find that the convexity property found in the classic cash flow management models continues to hold in our integrated cash balance model, but the optimal cash policy does not have the simple "borrow-up-to" and "invest-down-to" features as in the classic (L, U) policy. Instead, the optimal cash policy possesses the "non-borrow-up-to" and "non-invest-down-to" features, which resemble the "non-order-up-to" optimal policy found in the classic random yield problem (Henig and Gerchak 1990). The intuition is that the future cash balance in our problem depends on both the current cash level decision and the initial cash balance before adjustment, which is similar to the random yield problem where there is dependence between future inventory level and the initial inventory level before ordering.

We further derive qualitative insights about the value of payables finance. Our analysis reveals that it is the cash liquidity enabled by payables finance to hedge against cash flow uncertainty that generates value to the supplier, and such value increases with a lower payables finance interest rate and/or a higher payables finance amount. We also find that suppliers with higher short-term borrowing interest rates and/or higher cash flow uncertainty gain more value from the payables finance arrangement. These insights are consistent with industry reports that payables finance helps improve the financial sustainability of small suppliers by injecting additional liquidity into their trade finance needs (Wood 2019), and provide theoretical support for the wide adoption of payables finance for small suppliers with poor credit ratings and high cash flow uncertainty (PwC & Supply Chain Finance Community 2019). Moreover, the adoption of the blockchain technology in payables finance offers the promise to further improve the efficiency of invoices processing and reduce the risk of fraud, leading to a lower payables finance interest rate and a higher payables finance amount through wider adoption (Bain 2019). Thus, our model findings suggest that blockchain-based payables finance services are likely to offer greater value to small suppliers.

Regarding the equilibrium payment term extension for the buyer, we show that it is determined by the break even point between the value provided by payables finance and the cost increase to the supplier due to the delayed payment. The equilibrium payment extension increases as the payables finance interest rate decreases. Thus, the adoption of the blockchain technology, if it helps further lower the payables finance interest rate, may lead to longer payment extensions for the buyer in equilibrium. We also find that when the supplier's cash flow uncertainty is high, the buyer can ask for a longer payment extension. Interestingly, unlike the value of payables finance for the supplier, the equilibrium payment extension can be either increasing or decreasing in the payables finance amount, depending on the cash flow uncertainty the supplier faces. This result suggests that as the payables finance amount increases through wider adoption, the buyer may not necessarily always achieve longer payment extension in equilibrium (assuming that the payables finance interest rate remains unchanged).

Quantifying the value of payables finance and determining the equilibrium payment term extension both requires evaluating the optimal cash flow cost in our model. However, as discussed above, the "non-borrow-up-to" and "non-invest-down-to" features of the optimal cash policy greatly complicate the computation because one needs to compute the optimal cash levels for all beginning cash balance along all possible sample paths. To tackle this challenge, we derive an easy-to-compute heuristic policy termed the "approximate policy" based on an approximate dynamic program formulation, where the value function of the approximate dynamic program can serve as an easy-to-compute system cost lower bound for the original problem. This system cost lower bound can be further used in place of the original optimal cost function for performance evaluation purposes. We also derive the myopic policy for the original problem as a benchmark for comparison.

Our numerical study shows that both the approximate policy and the myopic policy achieve near-optimal performance in the original problem, with the approximate policy performing significantly better than the myopic policy across all experimental parameter scenarios. Given the strong performance of the approximate policy, we apply it along with the system cost lower bound to obtain lower bounds for the value of payables finance to the supplier and the equilibrium payment term extension for the buyer. Finally, we apply our model to a data set obtained from a major US chemicals company to estimate the value of payables finance to its suppliers and the equilibrium payment term extension it can achieve. Our analysis suggests that the payables finance arrangement can provide cost savings up to about 17% of the invoice amount for the firm's suppliers, and that the estimated payment extension for the firm ranges from 90 days to two years.

The rest of this paper is organized as follows. We review the related literature in §2. In §3, we present the random cash flow model for the supplier, derive the optimal cash policy, quantify the value of payables finance, and study the equilibrium payment term extension for the buyer. In §4 we study several heuristic policies for the problem and propose an easy-to-compute system cost lower bound. §5 contains our numerical studies with some real data applications, followed by §6 for concluding remarks. All proofs of our results are found in the Online Appendix.

2. Literature Review

Supply chain finance has been studied from various perspectives in the literature. Seifert and Seifert (2009) provided a high-level managerial assessment for supply chain finance. Randall and Farris (2009) used simple financial ratios to show the benefits of supply chain finance to all parties involved. Tanrisever et al. (2012) developed a model to show how supply chain finance influences the operational and financial decisions of supply chain parties. Using a single-period make-to-order and make-to-stock model, they determined the conditions under which the supply chain finance contract creates value for all parties. Tunca and Zhu (2018) considered a game-theoretical model for supplier finance and estimated the profits improvement with supplier finance based on empirical data from a Chinese online retailer. Their model assumes that the buyer sets the supplier finance interest rate without requesting payment extension and it does not consider cash flow uncertainty. Hu et al. (2018) considered another game-theoretical model involving supply chain finance, with a focus on the overall financing costs of the supply chain. In their model, the supplier pays the buyer (instead of the bank) an exogenous interest rate for an early withdrawal. More recently, Kouvelis and Xu (2021) developed a supply chain theory of factoring and reverse factoring to show when these post-shipment financing schemes should be adopted and who really benefits from the adoption. They considered a single-period setting and built upon the newsvendor financing model incorporating firms' credit ratings and liquidity risks. Babich and Hilary (2020) discussed various research opportunities in applying the blockchain technology to supply chain finance. Our paper contributes to this growing literature by providing solutions for quantifying the value of payables finance over multiple periods and the equilibrium payment term extension for the buyer.

A firm's future cash flow is usually difficult to forecast in practice (Gao 2018, Nallareddy et al. 2020), which motivates the random cash flow model considered in our paper. Our random cash flow model is closely related to the classic cash flow management literature (see Kallberg et al. 1982, for a review). In a seminal paper, Miller and Orr (1966) considered a discrete-time random walk model for firm's cash demand along with a two-parameter control limit policy. Eppen and Fama (1969) showed that the optimal cash management policy for a proportional cost model is an (L,U) threshold policy. Neave (1970) further extended the cash management model to include fixed transaction costs. An analysis of the continuous-time model with Brownian motion cash flow and fixed transaction costs can be found in Constantinides (1976) and Harrison (2013). Chen and Simchi-Levi (2009) showed that the concepts of symmetric K-convexity and (K,Q)-convexity developed in the inventory control literature can be used to characterize the optimal policy for the cash management model with fixed transaction cost. A common assumption in these models is that the cash balance evolves in a separate account without being affected by the interest gains and costs. Under this assumption, the cash management problem becomes mathematically equivalent to an inventory control problem. In fact, a more general state-dependent (L,U) threshold policy is shown to be optimal in inventory control problems with information updates for fashion products (Eppen and Iyer 1997) and spare parts systems (Chen et al. 2017). In our paper, by contrast, we consider a more realistic integrated cash balance model that allows all the interest gains and costs to accrue together with the cash balance in a single sum. We find the optimal cash policy possesses the "non-borrow-up-to" and "non-invest-down-to" features that resemble the "non-order-up-to" optimal policy found in the classic random yield problem (Henig and Gerchak 1990).

Our paper also contributes broadly to the stream of research on how financing arrangements affect supply chain operations. Gupta and Wang (2009) studied the impact of trade credit on supply chain contracting and inventory management. In a game-theoretical model, Kouvelis and Zhao (2018) presented a trade credit contract model where the interactions between the supplier and retailer are modeled as a Stackelberg game, and studied the impact of credit ratings on operational and financial decisions of a supply chain. Chen et al. (2021) studied the impact of trade credit on small businesses and their suppliers under a multiple-period setting, based on which they studied the expansion and inventory policies of the retailer under risk control. The payables finance arrangement considered in our paper differs from the trade credit or debt arrangements considered in the aforementioned papers. First, as described in the introduction, payables finance involves three parties, whereas trade credit or debt arrangement typically involves just two parties (such as a supplier and a buyer, or a supplier and a bank). Second, the supplier enjoys more flexibility regarding when and how much to receive the payment owed by the buyer under payables finance than under trade credit. Third, payables finance uses the supplier's invoice (receivables) as the collateral for financing, so the default risk is considerably lower than the bank lending directly to the supplier.

3. Model and Analysis

Consider a supply chain with a small supplier and a large buyer. The buyer, being larger and more solvent, enjoys a higher credit rating (or lower bank borrowing interest rate) than does the supplier. When the buyer orders from the supplier for goods worth W dollar value, under a common fixed-term arrangement the buyer can pay the supplier within N periods (e.g., days or weeks) after receiving the goods. In this case, the buyer, while having the option to pay the supplier sooner, will always pay on the due date because of the time value of money. To be clear, we refer to the due date N as the end of period N throughout the paper.

A payables finance arrangement is intended to replace the common fixed-term arrangement, with the aim to allow the supplier to receive the payment sooner while possibly further extending the payment due date for the buyer. According to Global Supply Chain Finance Forum (2020), the payables finance arrangement has the following sequence of events: (1) the supplier sends an invoice with amount W and due date N to the buyer after delivering goods or services; (2) the buyer approves the invoice and notifies the bank; (3) the supplier decides whether or not to request an early payment for a portion or all of W from the bank; (4) if an early payment is requested by the supplier, the bank will review and approve the requested amount, and also discount the amount with a predetermined interest rate ρ from the request date to the due date; and, finally, (5) on due date N (i.e., at the end of period N), the buyer pays the financed portion of W to the bank and the remaining portion (if any) to the supplier (see Figure 1 for a step-by-step illustration). We note that the payables finance service described above can be offered by either banks or blockchain-based FinTech platforms (De Meijer 2017). In fact, several major banks in China have recently launched their own blockchain-based payables finance platforms (Ledger Insights 2018b). For ease of reference, we shall simply refer to the payables finance service provider as "the bank" henceforth.

Let ρ_c denote the period-to-period discount interest rate, which can be viewed as the risk-free interest rate in each period (Li et al. 2013). The corresponding discount factor is denoted by

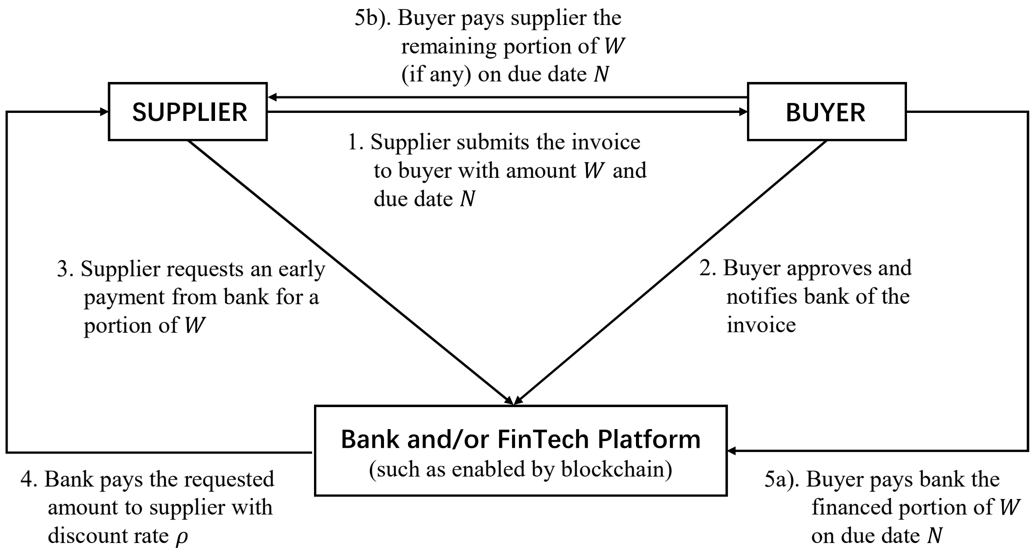


Figure 1 **Sequence of events under payables finance.**

$\delta = 1/(1+\rho_c)$. Both the buyer and the supplier have access to short-term loans, with borrowing interest rate ρ_i, with $i \in \{b, s\}$ (where "b" stands for buyer and "s" stands for supplier). We assume that $\rho_c < \rho_b < \rho_s$, reflecting the credit-rating dependent borrowing rates for the buyer and the supplier (see Kouvelis and Zhao 2018, for a similar assumption). Clearly, since the buyer is effectively "borrowing" the financed portion of W from the bank with the promise to repay the portion on due date N, the bank would charge an interest rate for payables finance with $\rho \geq \rho_b$. We shall assume this holds throughout the paper.

To keep things simple, consider a finite-horizon random cash flow problem for the supplier up to the payables finance payment due date N. The periods are numbered forward as $n = 1, 2, ..., N$, with period $N+1$ being the terminal period. For ease of exposition, we assume that the supplier faces an independent and identically distributed random cash flow ξ_n in each period n (we note that our analysis can be generalized to the case of independent but non-stationary cash flow). The random cash flow can be a result of the profit or loss from serving a random customer demand or the unexpected delayed payment from customers during a period. When $\xi_n > 0$, it represents a net cash *outflow*; and when $\xi_n < 0$, it represents a net cash *inflow*. The mean of the random cash flow in each period is denoted by $\mu = E[\xi_n] \leq 0$, so that the supplier on average has a negative net cash outflow, or, equivalently, a positive net cash inflow in each period. Also let $f(\cdot)$ and $F(\cdot)$ denote the probability density function (PDF) and cumulative distribution function (CDF) of ξ_n, respectively.

At the beginning of period n, the supplier has an initial on-hand cash balance x_n. Also let w_n be the net cash amount (after discounting) available to the supplier to draw from payables finance.

The initial balances are given by $x_1 = X$ and $w_1 = (1+\rho)^{-N}W$, where W is the initial payables finance amount expected on due date N.

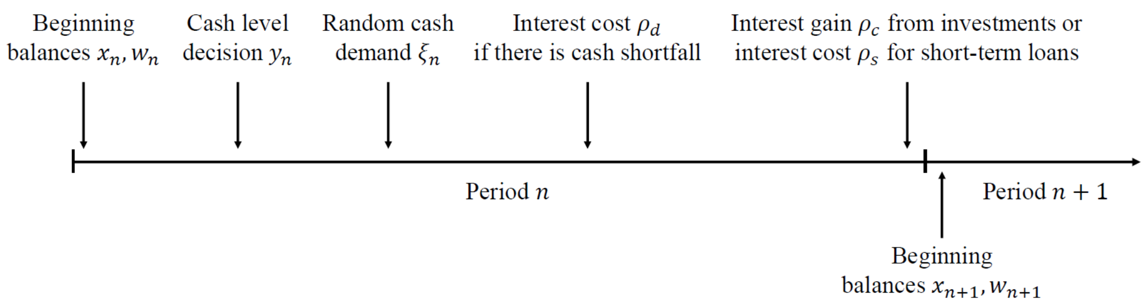


Figure 2 **Timeline for the supplier cash flow problem.**

Before the random cash flow is realized in a period n, the supplier determines a cash level y_n to meet the cash demand (see Figure 2 for an illustration). If the supplier chooses to keep a high cash level, it will use the available cash balance including the net cash amount w_n from payables finance and additional short-term loans if necessary. Note that, the maximum level that can be achieved without any additional short-term loans is $x_n + w_n$, where the supplier will withdraw all the net cash amount w_n from payables finance. To make the withdrawing decisions consistent with empirical observations in practice, we assume that the supplier's short-term loan interest rate is sufficiently high, such that it is optimal for the supplier to draw from payables finance before resorting to additional short-term loans (Esty et al. 2017, Tunca and Zhu 2018).[2]

On the other hand, if the supplier chooses to keep a cash level y_n below x_n, the excess amount $(x_n - y_n)$ is invested to earn a risk-free interest rate ρ_c. We assume that cash adjustment and investment are achieved immediately and there is no fixed transaction fee, reflecting the frictionless transactions enabled by the blockchain technology (Ledger Insights 2018a). The same assumption has also been adopted in the literature to enhance analytical tractability (Eppen and Fama 1969, Li et al. 2013).

The cash balance after netting the random cash flow in a period is $y_n - \xi_n$. If the balance is negative, the supplier needs to pay an extra interest cost $\rho_d(\xi_n - y_n)^+$, where $(x)^+ = \max\{x, 0\}$ and $\rho_d > \rho_s$ is the penalty interest rate for the negative ending balance before the supplier can act

[2] A condition that meets this assumption is that $\rho_s > \delta^{N-1}(1+\rho)^N - 1$, where the maximum present value of the would-be interest gain of payables finance is less than the present value of interest cost of using additional short-term loans. Note that, when $N = 1$, the condition reduces to $\rho_s > \rho$.

upon it at the beginning of the next period.[3] Thus, the supplier has incentive to borrow additional short-term loans at the beginning of a period to avoid the potential costly cash shortfall at the end of the period.

In the classic random cash flow literature (e.g., Eppen and Fama 1969, Neave 1970, Chen and Simchi-Levi 2009), a simplifying assumption was adopted such that the interest gains and costs are accrued and evaluated in a separate account from the cash balance itself. Such a decoupling assumption makes the cash flow problem more tractable in certain model settings. However, we find that it does not make the optimal policy in our payables finance problem more amenable for computation (see a detailed discussion in Appendix B). In what follows, we relax this decoupling assumption by considering a more realistic *integrated cash balance* model, that is, all interest gains and costs are allowed to accrue together with the cash balance in a single sum. We seek to reveal new insights about the cash flow management problem under this relaxed assumption.

3.1. Integrated Cash Balance Model

Under the integrated cash balance model, the on-hand cash balance in period $n+1$ (with $1 \le n \le N$) can be written as

$$x_{n+1} = y_n - \xi_n - \rho_d(\xi_n - y_n)^+ + (1+\rho_c)(x_n - y_n)^+ - (1+\rho_s)(y_n - x_n - w_n)^+, \tag{1}$$

where the third term is the extra interest cost paid to cover the temporary cash shortfall (if any), the fourth term is the risk-free interest return from investments (if any), and the last term is the interest cost of using additional short-term loans (if any) after the supplier withdraws all available amount from payables finance.[4] We note that the interest gains and costs in these last three terms are not accounted for in the on-hand cash balance transition under the decoupling assumption in the classic random cash flow literature.

Consider next the payables finance balance w_{n+1}. Because the balance is a discounted amount after netting the interest payment (with rate ρ) to the bank, the unused balance is effectively earning interest with rate ρ until due date N. Recall that $\rho_c < \rho$. This implies that the supplier would never draw from payables finance meanwhile investing money to earn risk-free interest. In other words, the supplier would draw from payables finance only if the desired cash level y_n exceeds the available cash balance x_n. Moreover, after withdrawing all the net cash amount w_n

[3] Alternatively, ρ_d can be also interpreted as the bankruptcy restructuring cost considered in Li et al. (2013). In our model, we do not consider the wipe-out bankruptcy because our focus is on short-term cash operations. We further note that the on-hand cash balance at the beginning of a period can be negative in our model, which represents money being borrowed/financed under the short-term loans.

[4] We note that the on-hand cash balance and payables finance balance are reviewed at the beginning of the terminal period $N+1$ before the remaining payables finance balance being transferred.

from payables finance in this period, the supplier could use additional short-term loans at a higher interest cost ρ_s, and the remaining payables finance balance w_{n+1} at the beginning of the next period becomes zero. As a result, we can write w_{n+1} (with $1 \le n \le N$) as

$$w_{n+1} = (1+\rho)\left[w_n - (y_n - x_n)^+\right]^+. \tag{2}$$

We refer the reader to Table A.1 in Appendix A for a list of the notations used in the paper.

The objective for the supplier is to maximize the discounted total accrued cash balance (including the payables finance balance) at the beginning of the terminal period $N+1$, which is given by

$$\Pi(X,W) = \max_{\{y_1,\dots,y_N\}} \delta^N E\left[x_{N+1} + w_{N+1}\right], \tag{3}$$

where $\{y_1, ..., y_N\}$ is the unconstrained cash level policy for each period before the payment due date N. With some term substitution and rearrangement, we can transform problem (3) into a cost minimization problem as follows (see the proof in Appendix A):

Proposition 1. *The following holds:*

$$\Pi(X,W) = X + \delta^N W - \frac{\delta - \delta^{N+1}}{1-\delta}\mu - \delta V_1(x_1, w_1), \tag{4}$$

where $V_1(x_1, w_1)$ is determined by the dynamic program: for $1 \le n \le N$,

$$\begin{aligned} V_n(x_n, w_n) =& \min_{y_n} \{G_n(y_n, x_n, w_n)\} \\ =& \min_{y_n} \Big\{\rho_c y_n + \rho_d E\left[(\xi_n - y_n)^+\right] + (\gamma_n(\rho) - \rho_c)\min\left\{(y_n - x_n)^+, w_n\right\} \\ & \qquad + (\rho_s - \rho_c)(y_n - x_n - w_n)^+ + \delta E\left[V_{n+1}(x_{n+1}, w_{n+1})\right]\Big\}, \\ V_{N+1}(\cdot,\cdot) =& 0, \end{aligned} \tag{5}$$

with $\gamma_n(\rho) = \delta^{N-n}(1+\rho)^{N-n+1} - 1$, $x_1 = X$, $w_1 = (1+\rho)^{-N} W$, and x_{n+1} and w_{n+1} given in (1) and (2), respectively. Moreover, $V_n(x_n, w_n)$ is decreasing in x_n and w_n.

Proposition 1 shows that the optimal discounted total cash balance $\Pi(X,W)$ consists of four parts: (1) the initial cash balance X, (2) the present value of the payables finance amount $\delta^N W$, (3) the present value of expected cash outflow $\sum_{i=1}^N \delta^i \mu$ over N periods, and (4) the discounted cost $\delta V_1(x_1, w_1)$ due to cash flow uncertainty. If there is no cash flow uncertainty, i.e., ξ_n is deterministic, then the supplier can set $y_n = 0$ in each period to achieve a minimum cost of $V_1(\cdot,\cdot) = 0$. In this case, the supplier would not make any early withdrawals from payables finance. On the other hand, if the cash demand ξ_n is random, an early withdrawal from payables finance can help reduce the cash flow cost V_1. Therefore, to understand and quantify the value of the payables finance arrangement, one needs to study the cost function V_1 under cash flow uncertainty.

The dynamic program objective function (5) contains five terms. The first term $\rho_c y_n$ is the opportunity cost (or the forgone risk-free interest) of setting a cash level $y_n \geq 0$ to meet the random cash flow.[5] The second term $\rho_d E[(\xi_n - y_n)^+]$ is the expected interest cost if the ending balance is negative in the period. The third term $(\gamma_n(\rho) - \rho_c)\min\{(y_n - x_n)^+, w_n\}$ is the opportunity cost of drawing the amount $\min\{(y_n - x_n)^+, w_n\}$ from payables finance, where $\gamma_n(\rho)$ is the would-be interest gain if the amount is kept unused until the due date (recall that the maximum amount in each period n that can be withdrawn from payables finance is w_n). The fourth term $(\rho_s - \rho_c)(y_n - x_n - w_n)^+$ is the extra interest cost of using additional short-term loans. Finally, the last term $\delta E[V_{n+1}(x_{n+1}, w_{n+1})]$ captures the (discounted) expected cost-to-go from period $n+1$ onward.

As shown in the proposition, the value function $V_n(x_n, w_n)$ also possesses some monotonicity property—it is decreasing in the cash balance x_n and the available amount w_n from payables amount. Intuitively, increasing the supplier's cash balance x_n in a period helps reduce the chance of early withdrawal from payables finance as well as using additional short-term loans, and also reduce the cost-to-go for future periods. Similarly, increasing the available payables finance balance w_n in a period helps reduce the chance of using additional short-term loans and also reduce the cost-to-go for future periods.

3.2. Optimal Cash Policies

According to the three respective decision cases of the dynamic program (5), i.e., $y_n \leq x_n$, $x_n < y_n \leq x_n + w_n$, and $y_n > x_n + w_n$, we can rewrite the objective function as follows.

$$G_n(y_n, x_n, w_n) = \begin{cases} G_n^U(y_n, x_n, w_n) & \text{if } y_n \leq x_n, \\ G_n^M(y_n, x_n, w_n) & \text{if } x_n < y_n \leq x_n + w_n, \\ G_n^L(y_n, x_n, w_n) & \text{if } y_n > x_n + w_n, \end{cases} \tag{6}$$

where

$$G_n^U(y_n, x_n, w_n) = \rho_c y_n + \rho_d E\left[(\xi_n - y_n)^+\right] + \delta H_n^U(y_n, x_n, w_n), \tag{7}$$

$$G_n^M(y_n, x_n, w_n) = \gamma_n(\rho) y_n - (\gamma_n(\rho) - \rho_c) x_n + \rho_d E\left[(\xi_n - y_n)^+\right] + \delta H_n^M(y_n, x_n, w_n), \tag{8}$$

$$G_n^L(y_n, x_n, w_n) = \rho_s y_n - (\rho_s - \rho_c) x_n - (\rho_s - \gamma_n(\rho)) w_n + \rho_d E\left[(\xi_n - y_n)^+\right] + \delta H_n^L(y_n, x_n, w_n), \tag{9}$$

with

$$H_n^U(y_n, x_n, w_n) = E\left[V_{n+1}((1+\rho_c)x_n - \rho_c y_n - \xi_n - \rho_d(\xi_n - y_n)^+, (1+\rho) w_n)\right], \tag{10}$$

$$H_n^M(y_n, x_n, w_n) = E\left[V_{n+1}(y_n - \xi_n - \rho_d(\xi_n - y_n)^+, (1+\rho)(w_n + x_n - y_n))\right], \tag{11}$$

[5] We note that it is possible that $y_n < 0$, e.g., when $x_n + w_n < 0$. In this case, there is an opportunity gain of $\rho_c y_n$ due to leverage, but there could also be a much higher interest cost ρ_d if the ending balance is negative in the period.

$$H_n^L(y_n, x_n, w_n) = E\left[V_{n+1}((1+\rho_s)(x_n+w_n) - \rho_s y_n - \xi_n - \rho_d(\xi_n - y_n)^+, 0)\right]. \tag{12}$$

We note that the expected cost-to-go functions $H_n^i(y_n, x_n, w_n)$ for $i \in \{U, M, L\}$ is defined according to the three decision cases $y_n \le x_n$, $x_n < y_n \le x_n + w_n$, and $y_n > x_n + w_n$, respectively. Therefore, the dynamic program (5) can be rewritten as a minimum of three cost minimization subproblems:

$$\begin{aligned}
&V_n(x_n, w_n)\\
=& \min_{y_n}\{G_n(y_n, x_n, w_n)\}\\
=& \min\left\{\min_{y_n \le x_n}\left\{G_n^U(y_n, x_n, w_n)\right\}, \min_{x_n < y_n \le x_n + w_n}\left\{G_n^M(y_n, x_n, w_n)\right\}, \min_{y_n > x_n + w_n}\left\{G_n^L(y_n, x_n, w_n)\right\}\right\}.
\end{aligned}$$

Furthermore, let $y_n^i(x_n, w_n)$ for $i \in \{U, M, L\}$ denote the *unconstrained* optimal solution to the three subproblems, respectively. That is, for $1 \le n \le N$, $i \in \{U, M, L\}$,

$$y_n^i(x_n, w_n) = \arg\min_{y_n}\{G_n^i(y_n, x_n, w_n)\}. \tag{13}$$

We note that $y_n^i(x_n, w_n)$ is a function of the beginning cash balances x_n and w_n. To see this, recall from the state transition equation (1), which can be written as

$$x_{n+1} = \begin{cases} (1+\rho_c)x_n - \rho_c y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } y_n \le x_n, \\ y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } x_n < y_n \le x_n + w_n, \\ (1+\rho_s)(x_n + w_n) - \rho_s y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } y_n > x_n + w_n. \end{cases} \tag{14}$$

It is clear that x_{n+1} is a function of y_n and x_n in all three cases. Similarly, it can be verified that $x_{n+1} + w_{n+1}$ is a function of y_n, x_n, and w_n in all three cases. This implies that the first-order derivative of $H_n^i(y_n, x_n, w_n)$ with respect to y_n is a function of x_n and w_n (the differentiability of $H_n^i(y_n, x_n, w_n)$ is verified in the proof of Proposition 2 in Appendix A). Hence, $y_n^i(x_n, w_n)$ is also a function of x_n and w_n due to its dependence on the derivative of $H_n^i(y_n, x_n, w_n)$.

By backward induction, we can first show the convexity of the three subproblem objective functions and the convexity at two kink points $y_n = x_n$ and $y_n = x_n + w_n$. Combining the convexity results leads to the convexity of the original objective function $G_n(y_n, x_n, w_n)$ in y_n, x_n, and w_n. This enables us to characterize the optimal cash policy (see the proof of Proposition 2 in Appendix A). The results are summarized in the following proposition:

Proposition 2. *For any $1 \le n \le N$, the following hold:*

(i) $G_n(y_n, x_n, w_n)$ is convex in y_n;

(ii) There exist three critical levels $L_n \le M_n \le U_n(w_n)$ (where L_n and M_n do not depend on x_n

and w_n, and $U_n(w_n)$ is weakly increasing in w_n), such that the optimal cash policy y_n^* for problem (5) is given by

$$y_n^* = \begin{cases} y_n^L(x_n, w_n) & \text{if } x_n + w_n < L_n, \\ x_n + w_n & \text{if } L_n \le x_n + w_n < M_n, \\ y_n^M(x_n, w_n) & \text{if } x_n < M_n \le x_n + w_n, \\ x_n & \text{if } M_n \le x_n < U_n(w_n), \\ y_n^U(x_n, w_n) & \text{if } x_n \ge U_n(w_n), \end{cases}$$

where $y_n^i(x_n, w_n)$ for $i \in \{U, M, L\}$ is defined in (13). Moreover, both $y_n^L(x_n, w_n)$ and $y_n^M(x_n, w_n)$ are functions of $x_n + w_n$ only.

Proposition 2 establishes the convexity of the dynamic program objective function $G_n(y_n, x_n, w_n)$ under the integrated cash balance model. The structure of the optimal cash policy features three critical levels $L_n \le M_n \le U_n(w_n)$ (their definitions are found in Appendix A). Figure 3 provides an illustration of the optimal policy. In each period n, first, if the available total cash balance $x_n + w_n$ (including the amount from payables finance) falls below a critical level L_n, the optimal decision is to draw the maximum available amount from payables finance, and borrow the difference $y_n^L(x_n, w_n) - x_n - w_n$ through additional short-term loans, so that the optimal cash level can be adjusted to $y_n^L(x_n, w_n)$. Second, if the available total cash balance $x_n + w_n$ falls in between the critical levels L_n and M_n, the optimal decision is to raise the cash level up to $x_n + w_n$, by drawing the maximum available amount from payables finance without using any additional short-term loans. Third, if the available total cash balance $x_n + w_n$ is above the critical level M_n while the on-hand cash balance x_n falls below M_n, the optimal decision is to draw the difference from payables finance, so that the optimal cash level can be adjusted to $y_n^M(x_n, w_n)$; no additional short-term loans are borrowed. Fourth, if the on-hand cash balance x_n falls in between the critical levels M_n and $U_n(w_n)$, the optimal cash level decision is to do nothing. Finally, if the on-hand cash balance x_n is above the critical level $U_n(w_n)$, the optimal decision is to lower the cash level to $y_n^U(x_n, w_n)$ and invest the difference to earn a risk-free interest; no withdrawal is made from payables finance. It is worth commenting that the critical level $U_n(w_n)$ is increasing in w_n. This implies that with a higher payables finance amount w_n, the supplier is less likely to invest the cash balance to earn a risk-free interest.

We shall highlight that the optimal cash policy for our problem distinguishes from the simple (L, U) policy in the classic cash flow literature (e.g., Eppen and Fama 1969) in the following aspects. First, obviously, the optimal cash policy has a new middle cash level $y_n^M(x_n, w_n)$ due to the access of payables finance. Second and more importantly, the optimal cash policy does not have the

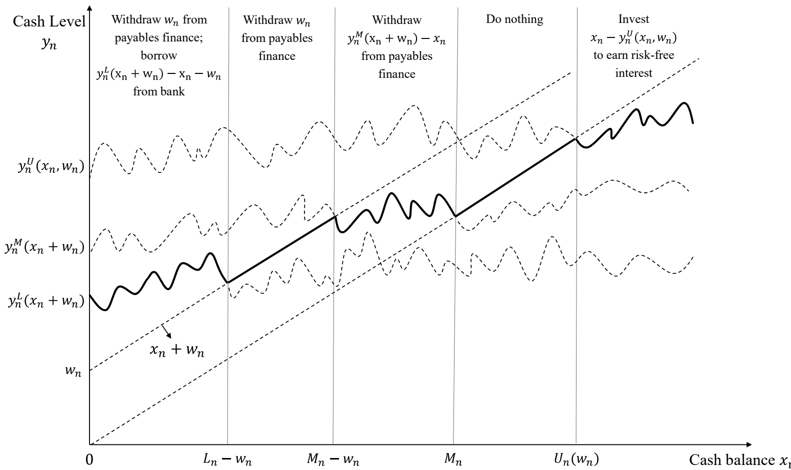


Figure 3 **Illustration of optimal cash policy.**

simple "borrow-up-to" and "invest-down-to" features as in the classic (L, U) policy. Instead, the optimal cash level does not necessarily equal the corresponding critical level, i.e., $y_n^L(x_n, w_n) \neq L_n$, $y_n^M(x_n, w_n) \neq M_n$, and $y_n^U(x_n, w_n) \neq U_n(w_n)$, which makes the optimal policy resemble the "non-order-up-to" optimal policy found in the classic random yield problem (Henig and Gerchak 1990). The intuition behind this result is that the future cash balance in our problem depends on both the current cash level decision and the initial cash balance before adjustment, which is similar to the random yield problem where there is dependence between future inventory level and the initial inventory level before ordering. This kind of "non-borrow-up-to" and "non-invest-down-to" features greatly complicates the computation of the optimal cash policy because one needs to compute the optimal cash levels for all beginning cash balance values.

One may wonder whether the "non-borrow-up-to" and "non-invest-down-to" features in the optimal policy might go away if the interest gains and costs are decoupled from the cash balance in our problem as in the classic cash flow literature. The answer turns out to be mixed (see Proposition B.2 in Appendix B). With the decoupling assumption, we can show that the lower and upper optimal cash levels will recover the classic (L, U) policy structure. However, the middle cash level still follows a "non-borrow-up-to" policy structure. Therefore, the resulting optimal cash policy is still challenging to compute. This shows that the complexity of our problem stems not only from our more realistic integrated cash balance model but also from the payables finance problem itself. In what follows, we derive some qualitative insights under the optimal cash policy for the value of payables finance in §3.3 and then study the equilibrium payment term extension for the buyer in §3.4.

3.3. Value of Payables Finance

According to the optimal value function (4), we can measure how payables finance helps reduce the supplier's cash flow cost. This cost reduction becomes the value of payables finance to the supplier. Consider first the case in which there is no payables finance arrangement. In this case, the initial payables finance amount $w_1 = 0$, but there is a scheduled cash amount W received on the payment due date N. Let $\Pi^0(X,W)$ denote the discounted total cash balance in this case, which can be written as

$$\Pi^0(X,W) = X + \delta^N W - \frac{\delta - \delta^{N+1}}{1-\delta}\mu - \delta V_1(X,0), \tag{15}$$

where $\delta^N W$ is the present value of the cash amount received on the payment due date N. Define the value of payables finance as $\Psi = \Pi(X,W) - \Pi^0(X,W)$. Further write out the Π and Π^0 functions as shown in Proposition 1 and the definition (15). The value of payables finance can be written as

$$\Psi = \delta V_1(X,0) - \delta V_1\left(X, (1+\rho)^{-N} W\right) \geq 0, \tag{16}$$

where the last inequality follows from the monotonicity of w_1 shown in Proposition 1. The following proposition summarizes properties of the value of payables finance:

PROPOSITION 3. *The following hold:*

(i) If cash demand ξ_n is deterministic, then $\Psi = 0$;

(ii) Ψ is decreasing in ρ and increasing in W.

Proposition 3(i) shows that without cash flow uncertainty, payables finance does not offer any value to the supplier. In other words, it is the cash liquidity enabled by payables finance to hedge cash flow uncertainty that generates value to the supplier. This insight is consistent with the empirical observation that payables finance helps improve the financial sustainability of small suppliers by injecting additional liquidity into their trade finance needs (Wood 2019). Proposition 3(ii) further shows that such value increases as the payables finance interest rate ρ decreases and/or as the payables finance amount W increases. Intuitively, decreasing ρ reduces the cost of withdrawing from payables finance, thus reducing the cash flow cost under payables finance. This leads to greater cash flow cost savings compared to the no payables finance arrangement. Moreover, having greater initial payables finance amount W allows the supplier to enjoy more cash liquidity of withdrawing from payables finance, thus generating greater value to the supplier. The adoption of the blockchain technology in payables finance offers the promise to further improve the efficiency of invoices processing and reduce the risk of fraud, leading to a lower payables finance interest rate and a higher payables finance amount through wider adoption (Bain 2019). Thus, our model findings suggest that blockchain-based payables finance services are likely to offer greater value to small suppliers.

To further understand how the level of cash flow uncertainty influences the value of payables finance, we consider a special case in which the payment due date $N = 1$ and the cash demand follows a normal distribution. Let $\Phi(\cdot)$ denote the standard normal CDF. Proposition 4 shows the analytical results for this case.

PROPOSITION 4. *Suppose $N = 1$ and the random cash flow follows a normal distribution (μ, σ). The following holds:*

(i) Ψ is increasing in ρ_s;

(ii) Suppose $\rho_d > 2\rho_s$. Then Ψ is increasing in σ. Moreover, $\Psi = 0$ if $\sigma \leq (X - \mu)/\Phi^{-1}\left(\frac{\rho_d - \rho}{\rho_d}\right)$; and $\Psi = \frac{\delta(\rho_s - \rho)}{1+\rho} W$ if $\sigma > \left(X - \mu + \frac{W}{1+\rho}\right)/\Phi^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)$.

Proposition 4(i) shows that the value of payables finance is increasing in the supplier's additional short-term loans borrowing rate ρ_s. When borrowing from additional short-term loans is costlier to the supplier, payables finance becomes more valuable because of its relatively low interest rate. In practice, small suppliers suffer from poor credit rating and have to pay high borrowing interest cost. This result provides a theoretical explanation for why payables finance is widely adopted for small suppliers located in emerging markets (PwC & Supply Chain Finance Community 2019).

Moreover, Proposition 4(ii) shows that when the interest cost of cash shortfall is relatively high and when payment due date is relatively short, the value of payables finance is increasing in the cash flow uncertainty level σ. Intuitively, as the cash flow uncertainty level increases, the supplier is more likely to withdraw all funds from payables finance to avoid the potential costly cash shortfall. Hence, payables finance becomes more valuable to the supplier. In addition, when the cash flow uncertainty is low, it is optimal for the supplier not to draw from payables finance before the payment due date. As a result, payables finance offers zero value to the supplier. On the other hand, when cash flow uncertainty is high (which is usually the case for small suppliers), it is optimal for the supplier to withdraw all payables finance immediately. In this case, the value of payables finance is essentially the interest cost saved from borrowing solely from the short-term loans. Therefore, the value of payables finance is proportional to the total amount W and interest difference $(\rho_s - \rho)$, and it does not depend on the supplier's initial on-hand cash balance X, as shown in the proposition.

3.4. Payment Term Extension

So far, we have studied the payables finance arrangement from the supplier's perspective, i.e., determining the supplier's optimal cash policy under payables finance as well as measuring the value of payables finance to the supplier. In this section, we explore the payables finance arrangement from the buyer's perspective. Specifically, we study the maximum payment due date extension that the buyer can achieve under payables finance while ensuring the supplier's participation based on the optimal value function V_1.

Clearly, a payment due date extension Δ reduces the present value of the payment for the supplier. As a result, the available cash amount for withdrawal from payables finance (after discounting) reduces from $w_1 = (1+\rho)^{-N}W$ to $w_1 = (1+\rho)^{-(N+\Delta)}W$. For simplicity, we assume that the supplier will withdraw all funds from payables finance on the original due date N. Therefore, the resulting discounted total cash balance is equivalent to $\Pi(X,(1+\rho)^{-\Delta}W)$. Define the value of payables finance with the extension Δ as

$$\Psi(\Delta) = \Pi(X,(1+\rho)^{-\Delta}W) - \Pi^0(X,W),$$

where $\Psi(0)$ is the same as Ψ defined in the previous section. Further write out the Π and Π^0 functions as shown in Proposition 1 and the definition (15). The value of payables finance with payment term extension can be written as

$$\begin{aligned}\Psi(\Delta) =& \delta\left[V_1(X,0) - V_1\left(X,(1+\rho)^{-(N+\Delta)}W\right)\right] - \frac{(1+\rho)^{\Delta}-1}{(1+\rho)^{\Delta}}\delta^N W \\ =& \Psi(0) - \delta\left[V_1\left(X,(1+\rho)^{-(N+\Delta)}W\right) - V_1\left(X,(1+\rho)^{-N}W\right)\right] - \frac{(1+\rho)^{\Delta}-1}{(1+\rho)^{\Delta}}\delta^N W. \end{aligned} \tag{17}$$

The last equality follows from the definition of (16). Observe that the value of payables finance $\Psi(\Delta)$ consists three parts: (1) the value of payables finance $\Psi(0)$ without any payment term extension, (2) the cost of $\delta\left[V_1\left(X,(1+\rho)^{-(N+\Delta)}W\right) - V_1\left(X,(1+\rho)^{-N}W\right)\right]$ due to reduced available payables finance for withdrawal as a result of payment extension, and (3) the loss of present value of $\frac{(1+\rho)^{\Delta}-1}{(1+\rho)^{\Delta}}\delta^N W$ due to payment extension. Note that, the last two parts capture the cost of payment term extension to the supplier. As the payment due date extension lengthens, the supplier's cost from receiving a delayed payment increases according to the monotonicity property shown in Proposition 1.

On the other hand, with a longer payment due date, the buyer could lower its working capital cost, leading to more efficient operations. Since payables finance works as a buyer-led program (see Global Supply Chain Finance Forum 2020), we can model the buyer as a leader and the bank and the suppliers as followers in a Stackelberg game. Specifically, the buyer can first work with the bank to set the payables finance interest rate at $\rho = \rho_b + r$, where r represents the bank's reservation interest premium for participating in the payables finance arrangement. With the payables finance interest rate ρ, the buyer can determine the payment term extension and make a take-it-or-leave-it offer to the supplier. The supplier then evaluates the value of the offer and decides whether or not to participate in the arrangement.

It is clear that given a payables finance interest rate ρ, the supplier is willing to participate in the payables finance arrangement with payment term extension Δ, as long as the value of payables finance $\Psi(\Delta) \geq 0$. The buyer's benefit is increasing in the payment extension Δ. Therefore, in equilibrium, given the payables finance interest rate ρ, the buyer would offer a payment term with the maximum due date extension as follows:

$$\Delta^* = \max\{\Delta \geq 0 \mid \text{s.t. } \Psi(\Delta) \geq 0\}. \tag{18}$$

According to the formulation (17), the condition in (18) becomes

$$\Psi(0) \geq \delta \left[V_1 \left(X, (1+\rho)^{-(N+\Delta)} W \right) - V_1 \left(X, (1+\rho)^{-N} W \right) \right] + \frac{(1+\rho)^{\Delta} - 1}{(1+\rho)^{\Delta}} \delta^N W. \tag{19}$$

Therefore, Δ^* is the point when the cash flow cost savings from payables finance without any payment term extension just break even with the cost increase due to the payment term extension. The following proposition summarizes the properties of the equilibrium payment extension:

Proposition 5. *The equilibrium payment extension Δ^* is decreasing in ρ. Suppose $N = 1$ and the random cash flow follows a normal distribution (μ, σ). Then Δ^* is increasing in σ if $\rho_d > 2\rho_s$; and Δ^* may be either increasing or decreasing in W, depending on the level of cash flow uncertainty.*

Proposition 5 shows that the equilibrium payment extension is decreasing in the payables finance interest rate ρ. Intuitively, as ρ decreases, the supplier gains more value from payables finance (see Proposition 3), which in turn allows the buyer to extend the payment due date further while still keeping the supplier participate. Thus, the adoption of the blockchain technology, if it helps further lower the payables finance interest rate as discussed in §3.3, may lead to longer payment extensions for the buyer in equilibrium.

Furthermore, when the interest cost of cash shortfall is relatively high and when payment due date is relatively short, the equilibrium payment extension is increasing in the cash flow uncertainty level σ. As shown in Proposition 4, the value of payables finance increases as σ increases, which enables a longer equilibrium payment extension for the buyer. In this case, we also find that the equilibrium payment extension can be either increasing or decreasing in the payables finance amount W, depending on cash flow uncertainty. Intuitively, increasing W would increase the loss of present value of the payment for the supplier due to the delayed payment, yet simultaneously increase the value of payables finance for the supplier. The first effect shortens the equilibrium payment term extension to ensure the supplier's participation, whereas the second effect allows for a longer equilibrium payment extension for the buyer. Because of this tradeoff, the buyer may not necessarily always achieve longer payment extension in equilibrium as the payables finance amount increases.

4. Heuristic Policies and Bounds

From our analysis in the previous section, evaluating the value of payables finance $\Psi(0)$ and the equilibrium payment extension Δ^* both require evaluating the optimal cost function V_1. Recall from the discussion of Proposition 2 that the optimal policy for our payables finance problem (5) is challenging to compute because one needs to determine $y_n^L(x_n, w_n)$, $y_n^M(x_n, w_n)$, and $y_n^U(x_n, w_n)$ for each possible x_n and w_n along all possible sample paths.

In this section, we derive an easy-to-compute heuristic policy based on an approximate dynamic program formulation, where the value function of the approximate dynamic program can also serve as an easy-to-compute system cost lower bound for the original problem. As a benchmark for comparison, we also derive the myopic policy for the original problem as another heuristic solution.

4.1. System Cost Lower Bound and Approximate Policy

Observe that the expression of x_{n+1} given in (1) can be rewritten as follows: for $1 \le n \le N$,

$$\begin{aligned} x_{n+1} = &(1+\rho)\left[x_n + \min\{(y_n - x_n)^+, w_n\} - \xi_n\right] \\ &\underbrace{-(\rho-\rho_c)(x_n - y_n)^+ - (\rho_s - \rho)(y_n - x_n - w_n)^+ - \rho(y_n - \xi_n)^+ - (\rho_d - \rho)(\xi_n - y_n)^+}_{\text{Terms to be dropped}}. \end{aligned}$$

Dropping the last four (negative) interest terms in x_{n+1} yields the following: for $1 \le n \le N$,

$$\begin{aligned} x_{n+1} &\approx \begin{cases} (1+\rho)(x_n - \xi_n) & \text{if } y_n \le x_n, \\ (1+\rho)(y_n - \xi_n) & \text{if } x_n < y_n \le x_n + w_n, \\ (1+\rho)(x_n + w_n - \xi_n) & \text{if } y_n > x_n + w_n, \end{cases} \\ x_{n+1} + w_{n+1} &\approx (1+\rho)(x_n + w_n - \xi_n), \end{aligned}$$

where, the righthand side is a function of either x_n, y_n, or $x_n + w_n$ in all cases. With this modification of the state trainsition, it can be shown that the first-order derivative of $E\left[V_{n+1}(x_{n+1}, w_{n+1})\right]$ with respect to y_n becomes independent of x_n and w_n, which helps remove the dependence of x_n and w_n in the optimal cash policy.

Therefore, we can define the following approximate cash balance transition recursively by dropping the terms as described above: for $1 \le n \le N$,

$$\tilde{x}_{n+1} = (1+\rho)\left[\tilde{x}_n + \min\{(y_n - \tilde{x}_n)^+, \tilde{w}_n\} - \xi_n\right], \tag{20}$$

$$\tilde{w}_{n+1} = (1+\rho)\left[\tilde{w}_n - (y_n - \tilde{x}_n)^+\right]^+, \tag{21}$$

where $\tilde{x}_1 = x_1 = X$ and $\tilde{w}_1 = w_1 = (1+\rho)^{-N} W$. With the modified state transition given in (20) and (21), we can define the following dynamic program based on $\tilde{x}_n$ and $\tilde{w}_n$: for $1 \le n \le N$,

$$\tilde{V}_n(\tilde{x}_n, \tilde{w}_n) = \min_{y_n}\left\{\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)\right\}$$

$$= \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c) \min\left\{ (y_n - \tilde{x}_n)^+, \tilde{w}_n \right\} \right.$$
$$\left. + (\rho_s - \rho_c)(y_n - \tilde{x}_n - \tilde{w}_n)^+ + \delta E\left[\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) \right] \right\}. \qquad (22)$$

The following proposition summarizes the property and optimal solution for the above problem:

PROPOSITION 6. *For any $1 \le n \le N$, the following hold:*

(i) $\tilde{V}_n(x_n, w_n) \le V_n(x_n, w_n)$ *for any given* x_n *and* w_n*;*

(ii) $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ *is convex in* y_n*;*

(iii) The optimal cash policy $y_n^\dagger$ *for problem (22) is given by*

$$y_n^\dagger = \begin{cases} L = F^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right) & \text{if } \tilde{x}_n + \tilde{w}_n < L, \\ \tilde{x}_n + \tilde{w}_n & \text{if } L \le \tilde{x}_n + \tilde{w}_n < \tilde{M}_n, \\ \tilde{M}_n & \text{if } \tilde{x}_n < \tilde{M}_n \le \tilde{x}_n + \tilde{w}_n, \\ \tilde{x}_n & \text{if } \tilde{M}_n \le \tilde{x}_n < U, \\ U = F^{-1}(\frac{\rho_d - \rho_c}{\rho_d}) & \text{if } \tilde{x}_n \ge U, \end{cases}$$

where

$$\tilde{M}_n = \arg\min_{y_n} \left\{ \gamma_n(\rho) y_n - (\gamma_n(\rho) - \rho_c)\tilde{x}_n + \rho_d E\left[(\xi_n - y_n)^+ \right] \right.$$
$$\left. + \delta E\left[\tilde{V}_{n+1}((1+\rho)(y_n - \xi_n), (1+\rho)(\tilde{w}_n + \tilde{x}_n - y_n)) \right] \right\},$$

with $L < \tilde{M}_n < U$*. Moreover,* $\tilde{M}_n \le \tilde{M}_{n+1}$ *for* $1 \le n < N$*.*

Proposition 6(i) shows that the value function of the modified problem (22) serves as a lower bound for that of the original problem (5) with the same initial cash balance. This property is quite useful for numerical evaluation of the performance of various heuristics (see §5 for details). The intuition of this result is as follows. Since $\tilde{x}_n$ is obtained by dropping negative terms in the original expressions of x_n recursively from the first period, it follows that $\tilde{x}_n \ge x_n$ and $\tilde{w}_n \ge w_n$. Recall from Proposition 1 that $V_n(x_n, w_n)$ is decreasing in x_n and w_n. Thus, it follows by induction that $\tilde{V}_n(x_n, w_n) \le V_n(x_n, w_n)$ for any given x_n and w_n.

Proposition 6(ii)-(iii) are proven together by backward induction. We refer to the optimal policy for this modified problem as the "approximate policy." The structure of the approximate policy is similar to that of the optimal policy in Proposition 2, where both policies feature three optimal cash levels. However, all three optimal cash levels under the policy equal their corresponding critical levels, and are independent of the current-period cash balances. Moreover, the "invest-down-to" level U and the "borrow-up-to" level L are determined by simple critical fractiles. Specifically, the critical level U for lowering cash level via investing is determined by a critical fractile $(\rho_d - \rho_c)/\rho_d$. This is analogous to the critical fractile in the inventory problem that balances the inventory overage and underage costs. In our cash flow problem, the unit overage cost of keeping too much cash is the forgone risk-free interest ρ_c. The unit underage cost of having too little cash is $\rho_d - \rho_c$ (with one unit less cash level, one has to pay the interest cost ρ_d for cash shortfall netting the risk-free interest ρ_c earned). Setting the marginal expected cash overage cost equal the marginal expected cash underage cost yields the above critical fractile. Similarly, the critical level L for raising cash level by borrowing additional short-term loans is based on the corresponding unit overage cost of ρ_s (which is the interest cost of obtaining additional short-term loans) and the corresponding unit underage cost $\rho_d - \rho_s$.

A few discussion points are in order. First, we note that the "invest-down-to" level U and the "borrow-up-to" level L under the approximate policy are the same as the upper and lower optimal cash levels when the interest gains and costs are decoupled from the cash balance (see Proposition B.2 in Appendix B). Second, the middle level $\tilde{M}_n$ is time-dependent and increasing in the period n, because $\gamma_n(\rho)$ is decreasing in time period n (see the expression in Proposition 1). This implies that under the approximate policy, the supplier is more likely to request withdrawals from payables finance in later periods. Intuitively, as it approaches to the due date (i.e., fewer periods are left before the due date), the would-be interest gain (if the amount is kept unused) for payables finance amount becomes smaller, and as a result, the supplier is more likely to draw from payables finance.

Finally, all three critical levels L, $\tilde{M}_n$, and U under the approximate policy are easy to compute, and as a result, we can replace the optimal cash levels of $y_n^L(x_n, w_n)$, $y_n^M(x_n, w_n)$, and $y_n^U(x_n, w_n)$ and the critical levels of L_n, M_n, and $U_n(w_n)$ in the original optimal policy with the approximate optimal levels of L, $\tilde{M}_n$, and U, respectively, to obtain a heuristic policy to our payables finance problem. This approximate policy is illustrated in Figure 4, where the dashed curves of $y_n^L(x_n, w_n)$, $y_n^M(x_n, w_n)$, and $y_n^U(x_n, w_n)$ are approximated by solid flat lines of L, $\tilde{M}_n$, and U.

4.2. Myopic Policy

As a benchmark for comparison, we derive the myopic policy for our payables finance problem (5). A myopic policy is the optimal cash level decision when the supplier only considers the current period cash flow cost.[6] The following proposition summarizes the myopic policy for our problem:

PROPOSITION 7. *For any $1 \le n \le N$, the myopic cash policy y_n^m for problem (5) is given by*

[6] The myopic policy structure is optimal in the single-period setting with $N = 1$. In the single-period problem, the supplier makes one-time cash level decision at the beginning of period one, so as to maximize the total cash balance in the terminal period. This single-period model setup is commonly used in the supply chain finance literature (see Tanrisever et al. 2012, Tunca and Zhu 2018, Kouvelis and Xu 2021).

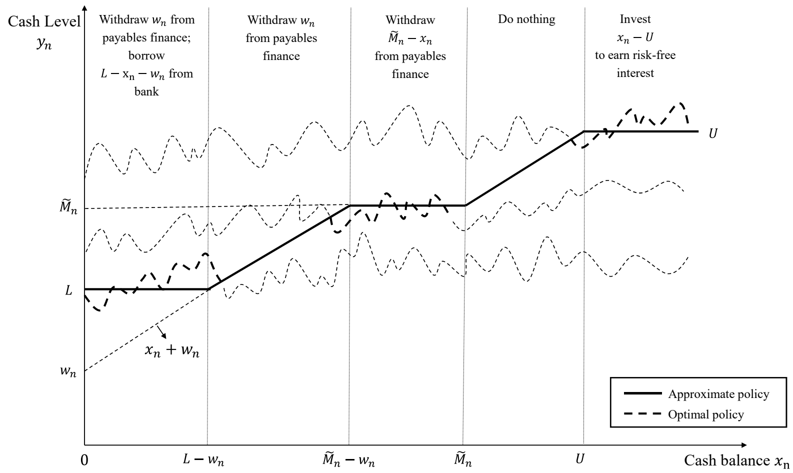


Figure 4 **Illustration of approximate cash policy with payables finance**

$$y_n^m = \begin{cases} L = F^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right) & \textit{if } x_n + w_n < L, \\ x_n + w_n & \textit{if } L \le x_n + w_n < M_n^m, \\ M_n^m = F^{-1}\left(\frac{\rho_d - \gamma_n(\rho)}{\rho_d}\right) & \textit{if } x_n < M_n^m \le x_n + w_n, \\ x_n & \textit{if } M_n^m \le x_n < U, \\ U = F^{-1}\left(\frac{\rho_d - \rho_c}{\rho_d}\right) & \textit{if } x_n \ge U, \end{cases}$$

with $L < M_n^m < U$*. Moreover,* $M_n^m \le \tilde{M}_n$ *and* $M_n^m \le M_{n+1}^m$ *for* $1 \le n < N$*.*

Proposition 7 shows that the myopic policy shares a similar strucuture to the approximate policy characterized in Proposition 6. Specifically, the "invest-down-to" level U and the "borrow-up-to" level L are the same as in the approximate policy. The only difference between the two policies is the critical level for raising cash level through payables finance, where the critical level M_n^m in the myopic policy is simply determined by a critical fractile. In determining M_n^m for raising cash level by drawing from payables finance, the corresponding unit overage cost is $\gamma_n(\rho)$ (which reflects the interest cost of drawing from payables finance) and the corresponding unit underage cost is $\rho_d - \gamma_n(\rho)$. Furthermore, the critical level M_n^m in the myopic policy is less than the critical level $\tilde{M}_n$ in the approximate policy. Intuitively, the myopic policy ignores the future cost of the current-period decision, so it may draw too little from the payables finance account compared to the approximate policy that takes into account the future cost. In addition, like in the approximate policy, M_n^m is also increasing in time period n due to the term $\gamma_n(\rho)$. Finally, it is also interesting to note that the myopic policy in the original problem is identical to the myopic policy in the modified problem (22). Like the approximate policy, we can also use the myopic policy as a heuristic policy to our payables finance problem.

5. Numerical Studies

In this section, we present two numerical studies for our problem. In the first study, we conduct numerical experiments to evaluate the easy-to-compute approximate and myopic policies specified in §4, demonstrating their near-optimal performance. In the second study, we use real cash flow data from a major US chemicals company to estimate the key parameters for our payables finance model and then evaluate the value of payables finance for the suppliers of this company. In addition, we estimate the maximum payment term extension this company can achieve by offering the payables finance arrangement to its suppliers from the data.

5.1. Evaluating Heuristic Policy Performance

In the first numerical study, we evaluate the performance of the approximate policy and the myopic policy. Let $V_1^z(x_1, w_1)$, with $z \in \{a, m\}$, denote the system cost by applying the approximate policy ("a") or the myopic policy ("m") to the original problem. The relative percentage of optimality gap can be defined as

$$\left(\frac{V_1^z(x_1, w_1)}{V_1(x_1, w_1)} - 1\right) \times 100\%, \quad z \in \{a, m\}.$$

As discussed in the previous section, the optimal system performance $V_1(x_1, w_1)$ is difficult to compute. To circumvent this challenge, we define an upper bound for the relative percentage of optimality gap as follows:

$$e^z := \left(\frac{V_1^z(x_1, w_1)}{\tilde{V}_1(x_1, w_1)} - 1\right) \times 100\% \geq \left(\frac{V_1^z(x_1, w_1)}{V_1(x_1, w_1)} - 1\right) \times 100\%, \quad z \in \{a, m\},$$

where the inequality follows from Proposition 6(i). We note that the value function of the lower bound system $\tilde{V}_1(x_1, w_1)$ is easy to compute. As a result, we shall use e^z to evaluate the optimality gap of the policy $z \in \{a, m\}$ throughout this section.

We assume one period consists of 10 days in this numerical study. In practice, a common fixed-term arrangement usually requires the buyer to pay within 30 to 60 days (Commercial Capital 2020). This corresponds to three to six periods in our model. Under the payables finance arrangement, the buyer can sometimes extend the payment due date much longer. Therefore, we allow the payment due dates to vary from three periods to 15 periods, i.e., $3 \leq N \leq 15$, in our numerical experiments.

We further assume the random cash flow follows a normal distribution with mean μ and standard deviation σ. The cash flow mean μ is normalized at one unit net cash inflow per period. For the rest of the model parameters, we vary them with three values, representing the low, medium, and high value scenarios. Specifically, we consider three cash flow uncertainty situations with $\sigma \in \{1, 3, 5\}$. For the initial cash balance, we set $X \in \{0, 1, 2\}$, representing zero cash balance, one times, and two times the expected net cash inflow per period. The initial payables finance amount W is chosen according to $W \in \{1,3,5\}$, representing one times, three times, and five times the expected net cash inflow per period.

For the risk-free interest rate, we use the Treasury Bill rate, which is about 2.2% annual interest rate. Dividing this annual rate by 36.5, we obtain the risk-free interest per period in our model to be about 0.06%. With the same reasoning, we set $\rho_c \in \{0.05\%, 0.06\%, 0.07\%\}$, corresponding to annual rates of about 1.8%, 2.2%, and 2.6%. The supplier's bank borrowing rate is significantly higher owing to its poor credit rating. We set $\rho_s \in \{0.4\%, 0.6\%, 0.8\%\}$, corresponding to annual rates of about 15%, 22%, and 29%. The payables finance interest rate is set as $\rho \in \{0.07\%, 0.08\%, 0.09\%\}$, corresponding to annual rates of about 2.6%, 2.9%, and 3.3%. Finally, According to Bankrate (2020), the average annual rate of emergency loans (for bad credit) is about 36%, corresponding to $\rho_d = 1\%$ in one period, which is fixed in all our experiments.

We vary one of the six model parameters at a time while keeping the rest five parameters at the medium value. Therefore, we have a total of $6 \times 3 = 18$ parameter scenarios. For each parameter scenario, we evaluate the optimality gap upper bounds e^a and e^m for different payment due dates with $3 \leq N \leq 15$. Table 1 reports the average and maximum optimality gap upper bounds of approximate and myopic policies under different parameter scenarios.

From the table, we first observe that both the approximate and myopic policies perform very well, achieving near-optimal performance. The largest optimality gap across all parameter scenarios is less than 4% in both policies. For the approximate policy, recall from (20) and (21) that only some of the interest terms (mostly are the interest difference terms) are dropped in the approximation. This suggests that, the approximation error appears to have a negligible impact on the system performance. Thus, the approximate policy performs significantly better than the myopic policy with its largest optimality gap 0.59%. We also expect that when the one-period interest rates become much larger, the approximate policy will perform considerably better than the myopic policy. Given the good performance and easy-to-compute feature of the approximate policy, we will use the approximate policy for our remaining numerical experiments. The optimality gap upper bounds of the approximate policy are plotted in Figure 5.

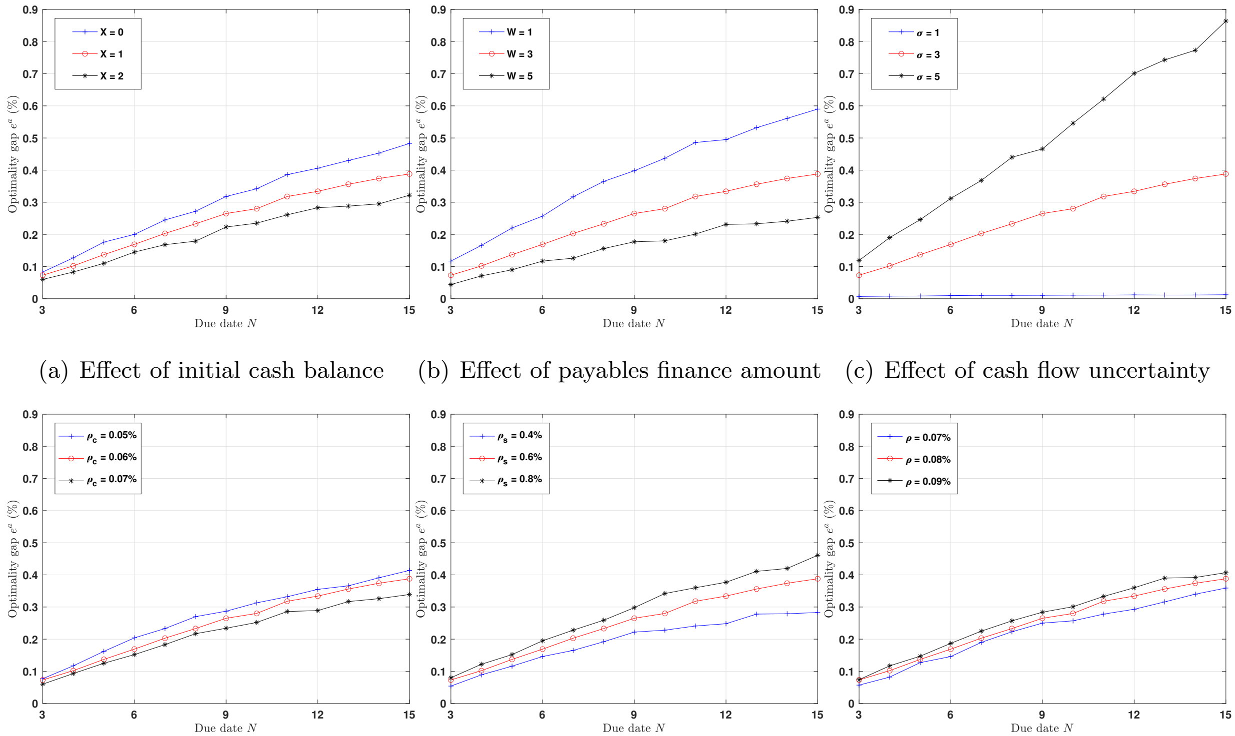
An interesting observation from the figure is that the optimality gap measure e^a increases in the due date N in all parameter scenarios, suggesting that the approximate policy performs better with a shorter due date. An intuitive explanation for this result is that the approximate policy approximates all three cash levels $y_n^L(x_n, w_n)$, $y_n^M(x_n, w_n)$, and $y_n^U(x_n, w_n)$ for each period n before the due date. Therefore, as the due date increases, the approximation error increases and the optimality gap widens.

In addition, Figure 5 (a)-(c) reveals that the optimality gap measure e^a is decreasing in the initial cash balance X and the payables finance amount W, and is increasing in the cash flow

Table 1 Optimality Gaps for Approximate Policy and Myopic Policy.

		Approximate Policy		Myopic Policy	
		Average (%)	Maximum (%)	Average (%)	Maximum (%)
X	0	0.302	0.483	1.969	3.945
	1	0.249	0.388	1.510	2.185
	2	0.204	0.322	0.766	1.192
W	1	0.380	0.590	1.076	1.809
	3	0.249	0.388	1.510	2.185
	5	0.163	0.253	1.725	2.540
σ	1	0.010	0.012	0.024	0.039
	3	0.249	0.388	1.510	2.185
	5	0.491	0.864	1.795	3.065
ρ_c	0.05%	0.271	0.414	1.968	2.746
	0.06%	0.249	0.388	1.510	2.185
	0.07%	0.221	0.339s	0.686	1.281
ρ_s	0.4%	0.195	0.283	1.502	2.266
	0.6%	0.249	0.388	1.510	2.185
	0.8%	0.285	0.461	1.522	2.310
ρ	0.07%	0.224	0.359	0.742	1.397
	0.08%	0.249	0.388	1.510	2.185
	0.09%	0.267	0.407	1.658	2.131

uncertainty σ. As the initial cash balance increases, the need for early withdrawal of payables finance reduces, leading to smaller approximation error. Similarly, as the payables finance amount increases, the approximation error also reduces. By contrast, as the cash flow uncertainty increases, the need for early withdrawal of payables finance increases, which increases the approximation error. Figure 5 (d)-(f) also reveals that the optimality gap measure e^a is decreasing in the risk-free interest rate ρ_c, and is increasing in the supplier's additional short-term loans borrowing rate ρ_s and the payables finance rate ρ. Increasing the risk-free interest rate helps reduce the incentive for early withdrawal of payables finance, so the approximation error reduces. On the other hand, increasing the supplier's additional short-term loans borrowing rate increases the need for early withdrawal of payables finance, so the approximation error increases. Additionally, increasing the payables finance rate increases those terms dropped in the approximation, thus leading to a greater



(a) Effect of initial cash balance (b) Effect of payables finance amount (c) Effect of cash flow uncertainty

(d) Effect of risk-free interest (e) Effect of supplier borrowing rate (f) Effect of payables finance rate

Figure 5 Optimality gap of the approximate policy with respect to various model parameters.

approximation error.

5.2. Applications with Real Data

To further test our model, we have collaborated with a major US chemicals company to obtain its supply chain cash flow data. The firm buys raw material from various suppliers and sells finished products to a number of global customers. We have collected detailed invoice-transaction-level accounts payable and accounts receivable data from the firm.

For this study, we focus on the transaction-level accounts payable data, which spans from October 24, 2017 to December 1, 2021. There are a total of 1,697 payment transactions with seven suppliers, covering 52 types of raw material. The payment terms with each supplier vary, depending on the raw material under transaction, with a wide range from 15 days to 120 days. The invoice amount varies from \$224 to \$8 million.

From the data, since a supplier might have different payment terms for different raw material, we group the raw material with the same payment terms for the supplier and treat it as a contract. This grouping strategy yields 11 contracts (we remove one supplier from the grouping due to sparsity of data). For each contract, we use the average invoice amount as the invoice amount W in our model, which ranges from \$6,298 to \$490,000 in the data. We further assume one period consists of 15 days according to the payment terms observed in the data. As such, the payment due dates of these contracts vary from one to eight periods in our model, i.e., $1 \le N \le 8$.

As in the first numerical study, we assume the supplier's cash flow follows a normal distribution. Since we do not have the supplier's cash flow data, we use the firm's payment flow to the supplier as a proxy with some adjustment. Specifically, we compute the standard deviation of the firm's aggregate payments to the supplier within a 15-day period and use it to approximate the supplier's cash *inflow* standard deviation, denoted by σ_{in}. We further assume the supplier faces an independent and identically distributed random cash *outflow* to its own suppliers. As a result, the net random cash flow for the supplier follows a normal distribution with mean $\mu = 0$ and $\sigma = \sqrt{2}\sigma_{in}$, where σ_{in} is found ranging from \$23,864 to \$4.77 million in the data. In addition, for ease of exposition, we assume the supplier's initial on-hand cash balance $X = 0$, with the understanding that as the supplier's initial on-hand cash balance increases, the cash flow cost for the supplier reduces (see Proposition 1). All interest rates used in this study are estimated to be 1.5 times the medium value used in the first numerical study because we assume one period consists of 15 days rather than 10 days in this second study.

5.2.1. Estimating Value of Payables Finance Recall from §3.3 that evaluating the value of payables finance requires solving the cash flow cost function V_1, which is difficult to compute. To address such challenge, we employ the approximate policy together with the system cost lower bound to compute the lower bound of the value of payables finance, which is given below:

$$\Psi^a := \delta\left[\tilde{V}_1(X,0) - V_1^a\left(X,(1+\rho)^{-N}W\right)\right] \le \delta\left[V_1(X,0) - V_1\left(X,(1+\rho)^{-N}W\right)\right].$$

The above inequality follows from the system cost lower bound property given in Proposition 6 and the fact that the approximate policy is sub-optimal for the original problem. We use Ψ^a to estimate the value of payables finance to each supplier based on the payment amount and due date.

Figure 6 illustrates the estimated value of payables finance for each of the 11 contracts in the data. Each bubble represents a contract, specified by a payment due date (on the x-axis), average payment amount W (on the y-axis), and an estimated supplier cash flow uncertainty σ in parenthesis next to the bubble. The size of the bubble represents the value of payables finance for the supplier, with the estimated dollar value shown next to the bubble.

We find that with the estimated model parameters, the value of payables finance for suppliers can range from \$679 to \$7,334, implying that the cost savings vary from 0.61% to 17.11% of the invoice amount in the contract. It is worth noting that this estimated value of payables finance is per invoice transaction. If the total number of invoice transactions during a year is taken into account, the annual cost savings from the payables finance arrangement can be quite sizable. Thus,

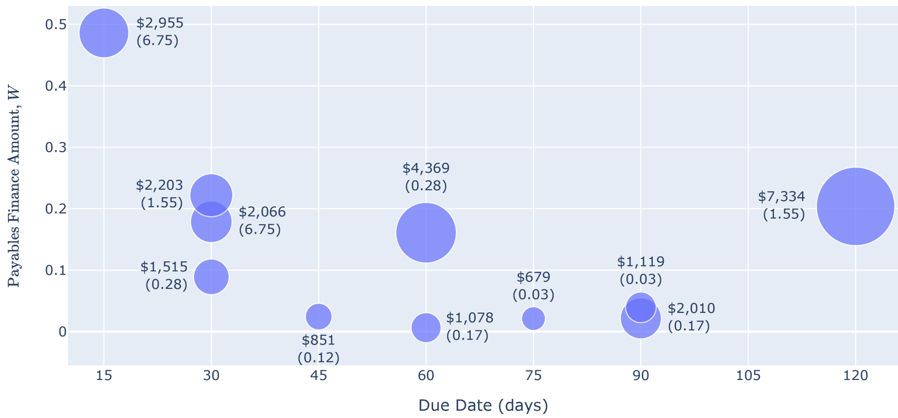


Figure 6 **Estimated value of payables finance per invoice transaction for the suppliers (σ and W are in millions, with σ shown in parenthesis).**

the firm can use our model to offer the payables finance arrangement to a selective set of contracts that have high annual cost savings potential for the suppliers.

Moreover, Figure 6 suggests that the highest payables finance value for the supplier arises when the payment due date N is the longest, while the lowest value appears when cash flow uncertainty σ is the lowest. Regarding the effect of payment due date N, without payables finance, the supplier who faces a longer payment due date is more likely to be in great need of cash to buffer for random cash flow, since it could only receive the payables finance amount at the payment due date. However, directly borrowing short-term loans can be costly to a small supplier with poor credit rating. As such, the existence of payables finance offers the supplier with a chance of receiving early payments at a more attractive rate. Therefore, such supplier with a longer payment due date can enjoy more benefits from payables finance. In addition, with a greater cash flow uncertainty, the supplier can enjoy more cash liquidity with payables finance. This also demonstrates that the analytical insights obtained in Proposition 4 are robust in more general cases. Furthermore, the estimated value of payables finance appears to increase as the payables finance amount increases. Such an effect is consistent with our model prediction given in Proposition 3.

5.2.2. Estimating Equilibrium Payment Term Extension Next, we apply the approximate policy together with the system cost lower bound to determine the lower bound of the equilibrium payment term extension, which can serve as a conservative estimate of the actual equilibrium payment term extension.

Given a payables finance interest rate, the equilibrium payment term extension that the buyer can achieve is determined by (18) in §3.4. From the condition for determining Δ^*, observe that

$$V_1(X,0) - V_1\left(X,(1+\rho)^{-(N+\Delta)}W\right) \geq \tilde{V}_1(X,0) - V_1^a\left(X,(1+\rho)^{-(N+\Delta)}W\right).$$

We can use the righthand side of the inequality (which is easy to compute) to replace the lefthand side used in condition (18) to obtain a lower bound Δ^a for Δ^*.

Figure 7 illustrates the payment due date extension Δ^a in the contracts. Each bubble in the figure represents a contract, with its current payment due date shown on the x-axis, the average payables finance amount W shown on the y-axis, and the estimated cash flow uncertainty σ shown in parenthesis next to the bubble. The size of the bubble represents the equilibrium payment term extension for the firm, which is also shown next to the bubble. In the estimation, we cap the maximum payment term extension to 2 years, i.e., 730 days.

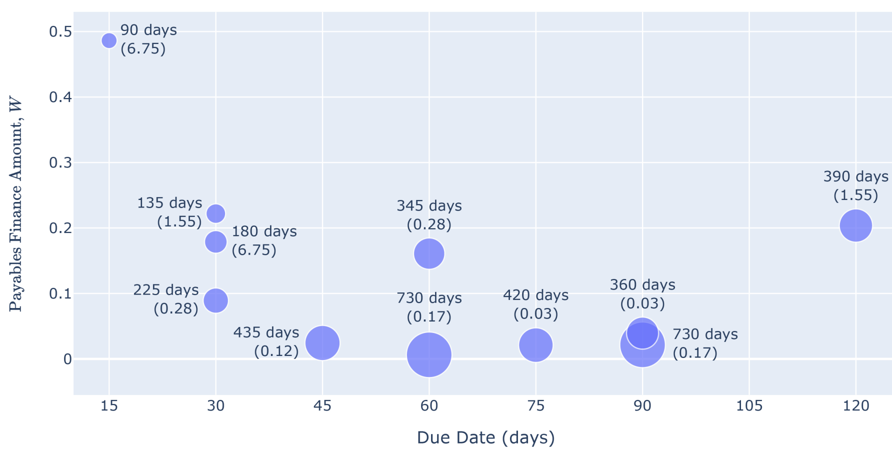


Figure 7 **Estimated payment term extension for the buyer (σ and W in millions, with σ shown in parenthesis).**

In Figure 7, the estimated payment term extension that can be achieved by the firm ranges from 90 days to two years. This shows strong evidence that, by offering the payables finance arrangement, the firm can extend its payment due date considerably to free up its working capital. In general, we observe from the figure that as the initial payment due date N increases, the buyer can ask for a longer payment due date extension. This is because with an initial long payment due date, the cost increase for the supplier due to additional payment extension is relatively small compared to the case with a short payment due date. Thus, our model can be used to gauge the parameters for payment term extension with the suppliers during the supply contract negotiation.

6. Conclusion

In this paper, we develop a random cash flow model to study the payables finance problem. Our integrated cash balance model extends the existing cash flow literature by allowing all interest gains and costs to accrue together with the cash balance in a single sum. We find the optimal cash policy for our problem does not have the simple "borrow-up-to" and "invest-down-to" features as in the classic (L, U) policy. Instead, the optimal cash policy possesses the "non-borrow-up-to" and "non-invest-down-to" features, which resemble the "non-order-up-to" optimal policy found in the classic random yield problem. With the optimal cash policy, we derive qualitative insights about the value of payables finance and study the equilibrium payment term extension for the buyer.

To tackle the computational challenge brought by the "non-borrow-up-to" and "non-invest-down-to" features of the optimal policy, we derive an easy-to-compute approximate policy based on an approximate dynamic program formulation, where the value function of the approximate dynamic program can serve as an easy-to-compute system cost lower bound for the original problem. We also derive the myopic policy for the original problem as a benchmark for comparison. Both the approximate policy and the myopic policy are shown to achieve near-optimal performance in the original problem, with the approximate policy performing significantly better than the myopic policy across all experimental parameter scenarios. We further demonstrate how to apply the approximate policy together with the system cost lower bound to obtain lower bounds for the value of payables finance to the supplier and the equilibrium payment term extension for the buyer, with applications to a data set obtained from a major US chemicals company.

As a first investigation into the value of payables finance with a model that accounts for the supplier cash flow uncertainty over multiple periods, we have restricted our attention to a single payables finance arrangement between the supplier and the buyer. This simpler case serves as a building block for the more complex case with multiple payables finance arrangements from one or many buyers. Based on the current analysis, we expect the interactions between multiple payables finance arrangements to pose some additional technical challenges, but may also yield additional valuable insights for practice.

Another exciting opportunity for future research is to consider blockchain-based payables finance in a multi-tier supply chain, referred to as "deep-tier supply chain finance" in practice (White 2020). Smaller suppliers at the further upstream of a supply chain are likely in greater need of cash liquidity enabled by payables finance. An interesting question for future work is to investigate and quantify the value of payables finance for such deep-tier suppliers. Although we expect some of our model insights to extend to the deep-tier setting, further work is necessary in order to confirm and improve our understanding of the cash flow management with deep-tier payables finance.

Finally, it is important to recognize that there may exist some information asymmetry among the parties involved in the payables finance arrangement. For example, the supplier's initial cash balance and the random cash demand distribution may not be readily observable by the buyer. The model presented in the current paper considers the full information case for the problem, which can be used as a benchmark for future analysis of the asymmetric information case. While the blockchain technology may help alleviate such information asymmetry problem to some extent, investigating the strategic interactions among the parties involved in payables finance can further expand our knowledge of the optimal design for payables finance.

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Online Appendix to "Optimal Cash Management with Payables Finance"

This online appendix contains two parts. Appendix A contains a list of notations used in the paper, all proofs for the results presented in the paper, as well as some auxiliary results together with their proofs. Appendix B contains the analysis of our payables finance problem under the decoupling cash balance assumption of the classic cash flow management literature.

A. Notations, Proofs, and Some Auxiliary Results

Table A.1 lists the notations used in the paper.

Table A.1 List of mathematical notations.

ρ_c	Risk-free interest rate
ρ_s	Supplier's bank borrowing (additional short-term loans) rate
ρ_b	Buyer's bank borrowing rate
ρ	Payables finance interest rate set by the bank
ρ_d	Interest cost for cash shortfall
δ	Period-to-period discount factor, with $\delta = 1/(1+\rho_c)$
N	Payables finance due date
X	Initial cash balance
W	Initial payables finance amount expected on due date N
ξ_n	Random cash demand during period n
x_n	On-hand cash balance available in period n
w_n	Cash amount available (after discounting) from payables finance in period n
y_n	Cash level used to meet random cash demand in period n

Proof of Proposition 1

First, we show that the problem (3) can be transformed into a cost minimization problem. According to the equation (3), the objective for the supplier is

$$\begin{aligned}\Pi(X,W) &= \max_{\{y_1,\ldots,y_N\}} \delta^N E\left[x_{N+1} + w_{N+1}\right] \\ &= \max_{\{y_1,\ldots,y_N\}} \left\{ x_1 + w_1 + \sum_{n=1}^{N} \delta^{n-1} E\left[\delta(x_{n+1}+w_{n+1}) - x_n - w_n\right] \right\}.\end{aligned}$$

Substituting the state transitions (1) and (2) into the term $\sum_{n=1}^{N}\delta^{n-1}E\left[\delta(x_{n+1}+w_{n+1})-x_n-w_n\right]$ yields

$$
\begin{aligned}
&\sum_{n=1}^{N}\delta^{n-1}E\left[\delta(x_{n+1}+w_{n+1})-x_n-w_n\right]\\
=&\sum_{n=1}^{N}\delta^{n-1}\Big\{\delta E\big[y_n-\xi_n-\rho_d(\xi_n-y_n)^+ +(1+\rho_c)(x_n-y_n)^+ -(1+\rho_s)(y_n-x_n-w_n)^+\\
&+(1+\rho)\left[w_n-(y_n-x_n)^+\right]^+\big]-x_n-w_n\Big\}\\
=&\sum_{n=1}^{N}\delta^{n-1}\left[-\delta\mu-\delta\rho_d E(\xi_n-y_n)^+ -(1-\delta)y_n-\delta(\rho-\rho_c)(y_n-x_n)^+ +\delta(\rho-\rho_c)w_n\right.\\
&\left.-\delta(\rho_s-\rho)(y_n-x_n-w_n)^+\right], \qquad \text{(A.1)}
\end{aligned}
$$

where the last equality is from the fact that $\left[w_n-(y_n-x_n)^+\right]^+ = w_n-(y_n-x_n)^+ +(y_n-x_n-w_n)^+$. According to (2), the payables finance balance w_{n+1} can be written as, for $1\le n\le N$,

$$
w_{n+1}=(1+\rho)\left[w_n-(y_n-x_n)^+\right]+(1+\rho)(y_n-x_n-w_n)^+.
$$

Plugging this into the term $\delta(\rho-\rho_c)w_n$ yields, for $1\le n\le N$,

$$
\begin{aligned}
&\delta(\rho-\rho_c)w_n\\
=&\delta(\rho-\rho_c)(1+\rho)w_{n-1}-\delta(\rho-\rho_c)(1+\rho)(y_{n-1}-x_{n-1})^+ +\delta(\rho-\rho_c)(1+\rho)(y_{n-1}-x_{n-1}-w_{n-1})^+\\
=&\cdots\\
=&\delta(\rho-\rho_c)(1+\rho)^{n-1}w_1-\delta(\rho-\rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i-x_i)^+ +\delta(\rho-\rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i-x_i-w_i)^+.
\end{aligned}
$$

Substituting the above expression back into equation (A.1), we have

$$
\begin{aligned}
&\sum_{n=1}^{N}\delta^{n-1}E\left[\delta(x_{n+1}+w_{n+1})-x_n-w_n\right]\\
=&\sum_{n=1}^{N}\delta^{n-1}\Bigg[-\delta\mu-\delta\rho_d E(\xi_n-y_n)^+ -(1-\delta)y_n-\delta(\rho-\rho_c)(y_n-x_n)^+ -\delta(\rho_s-\rho)(y_n-x_n-w_n)^+\\
&+\delta(\rho-\rho_c)(1+\rho)^{n-1}w_1-\delta(\rho-\rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}[(y_i-x_i)^+-(y_i-x_i-w_i)^+]\Bigg]\\
=&\sum_{n=1}^{N}\delta^{n}\Big\{-\mu-\rho_d E(\xi_n-y_n)^+ -\rho_c y_n-(1+\rho_c)\left(\left[\delta(1+\rho)\right]^{N-n+1}-1\right)\min\left\{(y_n-x_n)^+,w_n\right\}\\
&+(\rho_s-\rho_c)(y_n-x_n-w_n)^+\Big\}+\left(\left[\delta(1+\rho)\right]^{N}-1\right)w_1\\
=&-\delta(1-\delta)^{-1}\left(1-\delta^N\right)\mu+\left[\delta^N-\frac{1}{(1+\rho)^N}\right]W
\end{aligned}
$$

$$-\sum_{n=1}^{N}\delta^n\left\{\rho_c y_n+\rho_d E(\xi_n-y_n)^+ +(\gamma_n(\rho)-\rho_c)\min\left\{(y_n-x_n)^+,w_n\right\}+(\rho_s-\rho_c)(y_n-x_n-w_n)^+\right\},$$

where $\gamma_n(\rho)=\delta^{N-n}(1+\rho)^{N-n+1}-1$, representing the would-be interest gain if the amount is kept unused until the due date. By substituting the above expression back into equation (3) and rearranging it, we obtain the following equivalent cost minimization problem:

$$\begin{aligned}&\Pi(X,W)\\ =&X+\delta^N W-\frac{\delta-\delta^{N+1}}{1-\delta}\mu-\delta\min_{\{y_1,\cdots,y_N\}}\left\{\sum_{n=1}^{N}\delta^{n-1}\left\{\rho_c y_n+\rho_d E(\xi_n-y_n)^+\right.\right.\\ &\left.\left.+(\gamma_n(\rho)-\rho_c)\min\left\{(y_n-x_n)^+,w_n\right\}+(\rho_s-\rho_c)(y_n-x_n-w_n)^+\right\}\right\}.\end{aligned}$$

This can also be written as $\Pi(X,W)=X+\delta^N W-\frac{\delta-\delta^{N+1}}{1-\delta}\mu-\delta V_1(x_1,w_1)$, where $V_1(x_1,w_1)$ is determined by the following dynamic program: for $1\le n\le N$,

$$\begin{aligned}V_n(x_n,w_n)=\min_{y_n}&\left\{\rho_c y_n+\rho_d E(\xi_n-y_n)^+ +(\gamma_n(\rho)-\rho_c)\min\left\{(y_n-x_n)^+,w_n\right\}\right.\\ &\left.+(\rho_s-\rho_c)(y_n-x_n-w_n)^+ +\delta E\left[V_{n+1}(x_{n+1},w_{n+1})\right]\right\}.\end{aligned}$$

Next, we show that the supplier's cash flow cost $V_n(x_n,w_n)$ is decreasing in x_n and w_n for any $1\le n\le N$, using backward induction. We first rewrite the dynamic program as, for $1\le n\le N$,

$$\begin{aligned}V_n(x_n,w_n)=\min_{y_n}&\left\{\rho_c y_n+\rho_d E(\xi_n-y_n)^+ +(\gamma_n(\rho)-\rho_c)(y_n-x_n)^+ +(\rho_s-\gamma_n(\rho))(y_n-x_n-w_n)^+\right.\\ &\left.+\delta E\left[V_{n+1}(x_{n+1},w_{n+1})\right]\right\},\end{aligned}$$

where the equality is from the fact that $\min\{(y_n-x_n)^+,w_n\}=(y_n-x_n)^+-(y_n-x_n-w_n)^+$. Note that, we have assumed $\rho_s>\gamma_n(\rho)$ for all $1\le n\le N$ in the model setup. Since the terminal period $V_{N+1}(\cdot,\cdot)=0$, with any $\hat{x}_N\ge x_N$, we have

$$\begin{aligned}&V_N(x_N,w_N)\\ =&\min_{y_N}\left\{\rho_c y_N+\rho_d E(\xi_N-y_N)^+ +(\gamma_N(\rho)-\rho_c)(y_N-x_N)^+ +\left[\rho_s-\gamma_N(\rho)\right](y_N-x_N-w_N)^+\right\}\\ \ge&\min_{y_N}\left\{\rho_c y_N+\rho_d E(\xi_N-y_N)^+ +(\gamma_N(\rho)-\rho_c)(y_N-\hat{x}_N)^+ +\left[\rho_s-\gamma_N(\rho)\right](y_N-\hat{x}_N-w_N)^+\right\}\\ =&V_N(\hat{x}_N,w_N).\end{aligned}$$

Similarly, with any $\hat{w}_N\ge w_N$, we have

$$\begin{aligned}&V_N(x_N,w_N)\\ =&\min_{y_N}\left\{\rho_c y_N+\rho_d E(\xi_N-y_N)^+ +(\gamma_N(\rho)-\rho_c)(y_N-x_N)^+ +\left[\rho_s-\gamma_N(\rho)\right](y_N-x_N-w_N)^+\right\}\end{aligned}$$

$$\ge \min_{y_N} \left\{ \rho_c y_N + \rho_d E(\xi_N - y_N)^+ + (\gamma_N(\rho) - \rho_c)(y_N - x_N)^+ + [\rho_s - \gamma_N(\rho)] (y_N - x_N - \hat{w}_N)^+ \right\}$$
$$= V_N(x_N, \hat{w}_N).$$

Therefore, $V_N(x_N, w_N)$ is decreasing in x_N and w_N. Using backward induction, suppose that $V_{n+1}(x_{n+1}, w_{n+1})$ is decreasing in x_{n+1} and w_{n+1} for any $1 \le n < N-1$. With $\hat{x}_n \ge x_n$, according to the state transitions,

$$\begin{aligned}\hat{x}_{n+1} &= y_n - \xi_n - \rho_d(\xi_n - y_n)^+ + (1+\rho_c)(\hat{x}_n - y_n)^+ - (1+\rho_s)(y_n - \hat{x}_n - w_n)^+ \\ &\ge y_n - \xi_n - \rho_d(\xi_n - y_n)^+ + (1+\rho_c)(x_n - y_n)^+ - (1+\rho_s)(y_n - x_n - w_n)^+ = x_{n+1}, \\ \hat{w}_{n+1} &= (1+\rho) \left[w_n - (y_n - \hat{x}_n)^+ \right]^+ \ge (1+\rho) \left[w_n - (y_n - x_n)^+ \right]^+ = w_{n+1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}V_n(x_n, w_n) &= \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)(y_n - x_n)^+ + [\rho_s - \gamma_n(\rho)] (y_n - x_n - w_n)^+ \right. \\ &\quad \left. + \delta E \left[V_{n+1}(x_{n+1}, w_{n+1}) \right] \right\} \\ &\ge \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)(y_n - \hat{x}_n)^+ + [\rho_s - \gamma_n(\rho)] (y_n - \hat{x}_n - w_n)^+ \right. \\ &\quad \left. + \delta E \left[V_{n+1}(x_{n+1}, w_{n+1}) \right] \right\} \\ &\ge \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)(y_n - \hat{x}_n)^+ + [\rho_s - \gamma_n(\rho)] (y_n - \hat{x}_n - w_n)^+ \right. \\ &\quad \left. + \delta E \left[V_{n+1}(\hat{x}_{n+1}, \hat{w}_{n+1}) \right] \right\} = V_n(\hat{x}_n, w_n),\end{aligned}$$

where the last inequality follows from the backward induction assumption. Therefore, we conclude that $V_n(x_n, w_n)$ is decreasing in x_n for any $1 \le n \le N$. Similarly, with $\hat{w}_n \ge w_n$, according to the state transitions,

$$\begin{aligned}\hat{x}_{n+1} &= y_n - \xi_n - \rho_d(\xi_n - y_n)^+ + (1+\rho_c)(x_n - y_n)^+ - (1+\rho_s)(y_n - x_n - \hat{w}_n)^+ \\ &\ge y_n - \xi_n - \rho_d(\xi_n - y_n)^+ + (1+\rho_c)(x_n - y_n)^+ - (1+\rho_s)(y_n - x_n - w_n)^+ = x_{n+1}, \\ \hat{w}_{n+1} &= (1+\rho) \left[\hat{w}_n - (y_n - x_n)^+ \right]^+ \ge (1+\rho) \left[w_n - (y_n - x_n)^+ \right]^+ = w_{n+1}.\end{aligned}$$

Then, we have

$$\begin{aligned}V_n(x_n, w_n) &= \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)(y_n - x_n)^+ + [\rho_s - \gamma_n(\rho)] (y_n - x_n - w_n)^+ \right. \\ &\quad \left. + \delta E \left[V_{n+1}(x_{n+1}, w_{n+1}) \right] \right\} \\ &\ge \min_{y_n} \left\{ \rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)(y_n - x_n)^+ + [\rho_s - \gamma_n(\rho)] (y_n - x_n - \hat{w}_n)^+ \right. \\ &\quad \left. + \delta E \left[V_{n+1}(\hat{x}_{n+1}, \hat{w}_{n+1}) \right] \right\} = V_n(x_n, \hat{w}_n),\end{aligned}$$

where the inequality also follows from the backward induction assumption. Therefore, we conclude that $V_n(x_n, w_n)$ is decreasing in w_n for any $1 \le n \le N$. This completes the proof. $\square$

Proof of Proposition 2

We start by providing an overview of the backward induction proof. We first decompose the problem (5) into three cost minimization subproblems based on three decision cases: $y_n \le x_n$, $x_n < y_n \le x_n + w_n$, and $y_n > x_n + w_n$. By expressing the value function in period $n+1$ based on the five exhaustive cases according to the optimal policy in period $n+1$ (by the induction assumption), we prove the convexity of the three subproblem objective functions and the convexity at two kink points $y_n = x_n$ and $y_n = x_n + w_n$. Combining the convexity results leads to the convexity of the original objective function $G_n(y_n, x_n, w_n)$ in y_n, x_n, and w_n. This enables us to characterize the optimal cash policy in period n. We finally verify that $V_n(x_n, w_n)$ is convex and differentiable in x_n and w_n, which completes the backward induction.

Recall from the state transition equations (1) and (2), which can be written as:

$$x_{n+1} = \begin{cases} (1+\rho_c)x_n - \rho_c y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } y_n \le x_n, \\ y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } x_n < y_n \le x_n + w_n, \\ (1+\rho_s)(x_n + w_n) - \rho_s y_n - \xi_n - \rho_d(\xi_n - y_n)^+ & \text{if } y_n > x_n + w_n; \end{cases} \tag{A.2}$$

$$w_{n+1} = \begin{cases} (1+\rho)w_n & \text{if } y_n \le x_n, \\ (1+\rho)(w_n + x_n - y_n) & \text{if } x_n < y_n \le x_n + w_n, \\ 0 & \text{if } y_n > x_n + w_n. \end{cases} \tag{A.3}$$

For ease of notation, if no confusion arises, we drop the arguments y_n, x_n, and w_n in $G_n(y_n, x_n, w_n)$, $G_n^i(y_n, x_n, w_n)$, and $H_n^i(y_n, x_n, w_n)$ for $i \in \{U, M, L\}$. Moreover, we denote the first-order derivative of the function $\mathcal{X}(\cdot)$ with respect to θ as $\partial_\theta \mathcal{X} = \frac{\partial \mathcal{X}(\cdot)}{\partial \theta}$. In what follows, we use the backward induction to prove the following claims together: for $1 \le n \le N$,

(i) G_n^i for $i \in \{U, M, L\}$ is differentiable and convex in y_n, x_n, and w_n; G_n is convex in y_n, x_n, and w_n;

(ii) $\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)} \partial_{x_n} G_n^i \le \partial_{w_n} G_n^i \le \partial_{x_n} G_n^i + \gamma_{n-1}(\rho) - \rho_c$ for $i \in \{U, M, L\}$;

(iii) There exist three critical levels $L_n \le M_n \le U_n(w_n)$ (where L_n and M_n do not depend on x_n and w_n, and $U_n(w_n)$ depends only on w_n and is (weakly) increasing in w_n), such that the optimal cash policy y_n^* is given by

$$y_n^* = \begin{cases} y_n^L(x_n, w_n) & \text{if } x_n + w_n < L_n, \\ x_n + w_n & \text{if } L_n \le x_n + w_n < M_n, \\ y_n^M(x_n, w_n) & \text{if } x_n < M_n \le x_n + w_n, \\ x_n & \text{if } M_n \le x_n < U_n(w_n), \\ y_n^U(x_n, w_n) & \text{if } x_n \ge U_n(w_n), \end{cases}$$

where both $y_n^L(x_n, w_n)$ and $y_n^M(x_n, w_n)$ are functions of $x_n + w_n$ only;

(iv) $V_n(x_n, w_n)$ is convex and differentiable in x_n and w_n.

We first verify that these claims hold for the last period N. We then assume they hold for period $n+1$ and prove that they are also true for period n, completing the backward induction.

Verify the claims for period N. Because $V_{N+1}(\cdot,\cdot)=0$, we have $H_N^i = E[V_{N+1}(x_{N+1}, w_{N+1})] = 0$ for $i \in \{U, M, L\}$. It is straightforward to verify that G_N^i is differentiable and convex in y_N, x_N, and w_N. From (6), it remains to check the convexity of G_N at the two kink points: $y_N = x_N$ and $y_N = x_N + w_N$. At the kink point $y_N = x_N$, we have

$$\begin{aligned}\partial_{y_N} G_N \mid_{y_N \nearrow x_N} = \partial_{y_N} G_N^U \mid_{y_N \nearrow x_N} &= \rho_c - \rho_d + \rho_d F(x_N) \\ &< \gamma_N(\rho) - \rho_d + \rho_d F(x_N) \\ &= \partial_{y_N} G_N^M \mid_{y_N \searrow x_N} = \partial_{y_N} G_N \mid_{y_N \searrow x_N}.\end{aligned}$$

At the kink point $y_N = x_N + w_N$, we have

$$\begin{aligned}\partial_{y_N} G_N \mid_{y_N \nearrow (x_N+w_N)} = \partial_{y_N} G_N^M \mid_{y_N \nearrow (x_N+w_N)} &= \gamma_N(\rho) - \rho_d + \rho_d F(x_N + w_N) \\ &< \rho_s - \rho_d + \rho_d F(x_N + w_N) \\ &= \partial_{y_N} G_N^L \mid_{y_N \searrow (x_N+w_N)} = \partial_{y_N} G_N \mid_{y_N \searrow (x_N+w_N)}.\end{aligned}$$

Therefore, G_N is convex in y_N at these two kink points. Similarly, we can verify that G_N is convex in x_N and w_N at these two kink points too. It follows that G_N is convex in y_N, x_N, and w_N, so claim (i) holds for period N.

Because $H_N^i = 0$ and G_N^i is differentiable for $i \in \{U, M, L\}$, from definitions (7)-(9), we have $\partial_{x_N} G_N^U = \partial_{w_N} G_N^U = 0$, $\partial_{x_N} G_N^M = -\gamma_N(\rho) + \rho_c$, $\partial_{w_N} G_N^M = 0$, $\partial_{x_N} G_N^L = -\rho_s + \rho_c$ and $\partial_{w_N} G_N^M = -\rho_s + \gamma_N(\rho)$. It is thus straightforward to verify that claim (ii) holds for period N.

From the convexity of G_N^i in y_N and the definition in (13), the three optimal cash levels y_N^i for $i \in \{U, M, L\}$ are given uniquely by $y_N^U = U_N = F^{-1}\left(\frac{\rho_d - \rho_c}{\rho_d}\right)$, $y_N^M = M_N = F^{-1}\left(\frac{\rho_d - \rho}{\rho_d}\right)$, and $y_N^L = L_N = F^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)$, where the optimal cash levels are identical to the critical levels U_N, M_N, and L_N. In this case, the three critical levels U_N, M_N, and L_N do not depend on x_N and w_N. Because the cumulative distribution function $F(\cdot)$ is an increasing function, we have $L_N \le M_N \le U_N$ due to $\rho_c < \rho < \rho_s$. It follows that the optimal policy given by claim (iii) holds for period N.

As a result, we can write the value function $V_N(x_N, w_N)$ based on the optimal policy as

$$V_N(x_N, w_N) = \begin{cases} G_N^L(L_N, x_N, w_N) & \text{if } x_N + w_N < L_N, \\ G_N^M(x_N + w_N, x_N, w_N) & \text{if } L_N \le x_N + w_N < M_N, \\ G_N^M(M_N, x_N, w_N) & \text{if } x_N < M_N \le x_N + w_N, \\ G_N^U(x_N, x_N, w_N) & \text{if } M_N \le x_N < U_N, \\ G_N^U(U_N, x_N, w_N) & \text{if } x_N \ge U_N. \end{cases}$$

According to Theorem 5.7 of Rockafellar (1970), the convexity of $G_N(y_N, x_N, w_N)$ in x_N and w_N implies the convexity of $V_N(x_N, w_N)$ in x_N and w_N. To check the differentiability of $V_N(x_N, w_N)$ in x_N and w_N, recall that G_N^i for $i \in \{U, M, L\}$ is differentiable in x_N and w_N. By applying the envelope theorem to each of the five segments of $V_N(x_N, w_N)$ as well as recognizing that G_N is differentiable in x_N and w_N at the four connecting points: $x_N + w_N = L_N$, $x_N + w_N = M_N$, $x_N = M_N$, and $x_N = U_N$, it follows that $V_N(x_N, w_N)$ is differentiable in x_N and w_N. This verifies claim (iv) for period N.

Assume the claims hold for period $n+1$ and verify them for period n. Because $V_{n+1}(x_{n+1}, w_{n+1})$ is differentiable in x_{n+1} and w_{n+1} by the induction assumption of claim (iv), from definitions (10)-(12), it follows that $H_n^i(y_n, x_n, w_n)$ for $i \in \{U, M, L\}$ is also differentiable in y_n, x_n, and w_n. This implies that G_n^i for $i \in \{U, M, L\}$ is differentiable in y_n, x_n, and w_n from definitions (7)-(9).

Next, we prove the convexity of $G_n(y_n, x_n, w_n)$ in y_n, x_n, and w_n by first showing convexity of the three subproblems $G_n^i(y_n, x_n, w_n)$ for $i = \{U, M, L\}$ and then verifying convexity at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$. Given ξ_n, the cash balance transitions x_{n+1} and w_{n+1} are linear in x_n, w_n and y_n in the three decision cases (see (A.2) and (A.3)). According to Theorem 5.7 of Rockafellar (1970), the convexity of $V_{n+1}(x_{n+1}, w_{n+1})$ in x_{n+1} and w_{n+1} implies the convexity of $V_{n+1}(x_{n+1}, w_{n+1})$ in y_n, x_n, and w_n in the three cases given ξ_n. Taking the expectation over ξ_n, we have $E[V_{n+1}(x_{n+1}, w_{n+1})]$ is also convex in y_n, x_n, and w_n in the three cases. Therefore, $H_n^i(y_n, x_n, w_n)$ for $i \in \{U, M, L\}$ is convex in y_n, x_n, and w_n. This implies that $G_n^i(y_n, x_n, w_n)$ for $i \in \{U, M, L\}$ is also convex in y_n, x_n, and w_n. It remains to check the convexity of G_n at the two kink points: $y_n = x_n$ and $y_n = x_n + w_n$.

From the induction assumption of claim (iii), we first write the following five exhaustive cases for the value function $V_{n+1}(x_{n+1}, w_{n+1})$. If no confusion arises, we drop the arguments x_{n+1} and w_{n+1} from $y_{n+1}^i(x_{n+1}, w_{n+1})$ for $i \in \{U, M, L\}$, for ease of notation.

(1) if $x_{n+1} + w_{n+1} < L_{n+1}$,

$$\begin{aligned} V_{n+1}(x_{n+1}, w_{n+1}) = &\rho_s y_{n+1}^L - (\rho_s - \rho_c)x_{n+1} - (\rho_s - \gamma_{n+1}(\rho))w_{n+1} \\ &+ \rho_d E\left[(\xi_{n+1} - y_{n+1}^L)^+\right] + \delta H_{n+1}^L(y_{n+1}^L, x_{n+1}, w_{n+1}); \end{aligned}$$

(2) if $L_{n+1} \le x_{n+1} + w_{n+1} < M_{n+1}$,

$$\begin{aligned} V_{n+1}(x_{n+1}, w_{n+1}) = &\rho_c x_{n+1} + \gamma_{n+1}(\rho)w_{n+1} + \rho_d E\left[(\xi_{n+1} - x_{n+1} - w_{n+1})^+\right] \\ &+ \delta H_{n+1}^M(x_{n+1} + w_{n+1}, x_{n+1}, w_{n+1}); \end{aligned}$$

(3) if $x_{n+1} < M_{n+1} \le x_{n+1} + w_{n+1}$,

$$\begin{aligned} V_{n+1}(x_{n+1}, w_{n+1}) = &\rho_c y_{n+1}^M + (\gamma_{n+1}(\rho) - \rho_c)(y_{n+1}^M - x_{n+1}) \\ &+ \rho_d E\left[(\xi_{n+1} - y_{n+1}^M)^+\right] + \delta H_{n+1}^M(y_{n+1}^M, x_{n+1}, w_{n+1}); \end{aligned}$$

(4) if $M_{n+1} \le x_{n+1} < U_{n+1}(w_{n+1})$,

$$V_{n+1}(x_{n+1}, w_{n+1}) = \rho_c x_{n+1} + \rho_d E\left[(\xi_{n+1} - x_{n+1})^+\right] + \delta H^U_{n+1}(x_{n+1}, x_{n+1}, w_{n+1});$$

(5) if $x_{n+1} \ge U_{n+1}(w_{n+1})$,

$$V_{n+1}(x_{n+1}, w_{n+1}) = \rho_c y^U_{n+1} + \rho_d E\left[(\xi_{n+1} - y^U_{n+1})^+\right] + \delta H^U_{n+1}(y^U_{n+1}, x_{n+1}, w_{n+1}).$$

Recall from Proposition 1 that $V_{n+2}(x_{n+2}, w_{n+2})$ is decreasing in x_{n+2} and w_{n+2}. Because x_{n+2} and w_{n+2} are increasing x_{n+1} and w_{n+1} (see (1) and (2)), it follows that $V_{n+2}(x_{n+2}, w_{n+2})$ is decreasing in x_{n+1} and w_{n+1}. Therefore, $E[V_{n+2}(x_{n+2}, w_{n+2})]$ and $G^i_{n+1}, i \in \{U, M, L\}$, are decreasing in x_{n+1} and w_{n+1}. In other words, $\partial_{x_{n+1}} G^i_{n+1} \le 0$ and $\partial_{w_{n+1}} G^i_{n+1} \le 0$ for $i \in \{U, M, L\}$. We will utilize this property to prove the convexity of G_n at the two kink points. We note that given ξ_n, at the kink point $y_n = x_n$, G^U_n and G^M_n share the cash balances x_{n+1} and w_{n+1}; at the kink point $y_n = x_n + w_n$, G^M_n and G^L_n share the cash balances x_{n+1} and w_{n+1}. Based on the five exhaustive cases of $V_{n+1}(x_{n+1}, w_{n+1})$ above, we obtain the following:

(1) For any ξ_n such that $x_{n+1} + w_{n+1} < L_{n+1}$: First, with respect to y_n, when $\xi_n < y_n$, we have

$$\begin{aligned}
&\partial_{y_n} H^U_n = -\rho_c(\delta\partial_{x_{n+1}} H^L_{n+1} - \rho_s + \rho_c),\\
&\partial_{y_n} H^M_n = (\delta\partial_{x_{n+1}} H^L_{n+1} - \rho_s + \rho_c) - (1+\rho)(\delta\partial_{w_{n+1}} H^L_{n+1} - \rho_s + \gamma_{n+1}(\rho)),\\
&\partial_{y_n} H^L_n = -\rho_s(\delta\partial_{x_{n+1}} H^L_{n+1} - \rho_s + \rho_c).
\end{aligned}$$

At the kink point $y_n = x_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n \mid_{y_n \searrow x_n}\\
=&\partial_{y_n} G^U_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G^M_n \mid_{y_n \searrow x_n}\\
=&\partial_{y_n} H^U_n + (1+\rho_c)\rho_c - \left[\partial_{y_n} H^M_n + (1+\rho_c)\gamma_n(\rho)\right]\\
=&(1+\rho)(\delta\partial_{w_{n+1}} H^L_{n+1} - \rho_s + \gamma_{n+1}(\rho)) - (1+\rho_c)(\delta\partial_{x_{n+1}} H^L_{n+1} - \rho_s + \rho_c) - (1+\rho_c)(\gamma_n(\rho) - \rho_c)\\
=&(1+\rho)\partial_{w_{n+1}} G^L_{n+1} - (1+\rho_c)(\partial_{x_{n+1}} G^L_{n+1} + \gamma_n(\rho) - \rho_c)\\
\le&(\rho - \rho_c)\partial_{w_{n+1}} G^L_{n+1} \le 0,
\end{aligned}$$

where the first inequality follows from the induction assumption of claim (ii) and the second inequality follows from $\partial_{w_{n+1}} G^L_{n+1} \le 0$. At the kink point $y_n = x_n + w_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow (x_n + w_n)} - \partial_{y_n} G_n \mid_{y_n \searrow (x_n + w_n)}\\
=&\partial_{y_n} G^M_n \mid_{y_n \nearrow (x_n + w_n)} - \partial_{y_n} G^L_n \mid_{y_n \searrow (x_n + w_n)}\\
=&\partial_{y_n} H^M_n + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n} H^L_n + (1+\rho_c)\rho_s\right]
\end{aligned}$$

$$
\begin{aligned}
=&(1+\rho_s)(\delta\partial_{x_{n+1}}H_{n+1}^L-\rho_s+\rho_c)-(1+\rho)(\delta\partial_{w_{n+1}}H_{n+1}^L-\rho_s+\gamma_{n+1}(\rho))-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
=&(1+\rho_s)\partial_{x_{n+1}}G_{n+1}^L-(1+\rho)\partial_{w_{n+1}}G_{n+1}^L-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
\le&(1+\rho_s)\partial_{x_{n+1}}G_{n+1}^L-(1+\gamma_{N-n}(\rho))\partial_{w_{n+1}}G_{n+1}^L-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
=&(1+\gamma_{N-n}(\rho))\left(\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}}G_{n+1}^L-\partial_{w_{n+1}}G_{n+1}^L\right)-(1+\rho_c)(\rho_s-\gamma_n(\rho))\le 0,
\end{aligned}
$$

where the first inequality follows from $\gamma_{N-n}(\rho)\ge\gamma_N(\rho)=\rho$ and $\partial_{w_{n+1}}G_{n+1}^L\le 0$, and the last inequality follows from the induction assumption of claim (ii). Therefore, G_n is convex in y_n at the two kink points. Similarly, when $\xi_n\ge y_n$, it can be verified that G_n is convex in y_n at the two kink points.

Second, with respect to x_n, we have

$$
\begin{aligned}
&\partial_{x_n}H_n^U=(1+\rho_c)(\delta\partial_{x_{n+1}}H_{n+1}^L-\rho_s+\rho_c)=(1+\rho_c)\partial_{x_{n+1}}G_{n+1}^L,\\
&\partial_{x_n}H_n^M=(1+\rho)(\delta\partial_{w_{n+1}}H_{n+1}^L-\rho_s+\gamma_{n+1}(\rho)),\\
&\partial_{x_n}H_n^L=(1+\rho_s)(\delta\partial_{x_{n+1}}H_{n+1}^L-\rho_s+\rho_c).
\end{aligned}
\tag{A.4}
$$

By using the induction assumption of claim (ii) and the fact that $\partial_{w_{n+1}}G_{n+1}^L\le 0$, it can be verified that G_n is convex in x_n at the two kink points.

Finally, with respect to w_n, we have

$$
\begin{aligned}
&\partial_{w_n}H_n^U=\partial_{w_n}H_n^M=(1+\rho)(\delta\partial_{w_{n+1}}H_{n+1}^L-\rho_s+\gamma_{n+1}(\rho))=(1+\rho)\partial_{w_{n+1}}G_{n+1}^L,\\
&\partial_{w_n}H_n^L=(1+\rho_s)(\delta\partial_{x_{n+1}}H_{n+1}^L-\rho_s+\rho_c).
\end{aligned}
\tag{A.5}
$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_{w_{n+1}}G_{n+1}^L\le 0$, it can be verified that G_n is convex in w_n at the two kink points. Therefore, we have verified that $G_n(y_n,x_n,w_n)$ is convex in y_n, x_n, and w_n at the two kink points $y_n=x_n$ and $y_n=x_n+w_n$.

(2) For any ξ_n such that $L_{n+1}\le x_{n+1}+w_{n+1}<M_{n+1}$: First, with respect to y_n, when $\xi_n<y_n$,

$$
\begin{aligned}
&\partial_{y_n}H_n^U=-\rho_c\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}}-\rho_c(\delta\partial_{x_{n+1}}H_{n+1}^M-\gamma_{n+1}(\rho)+\rho_c),\\
&\partial_{y_n}H_n^M=-\rho\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}}+(\delta\partial_{x_{n+1}}H_{n+1}^M-\gamma_{n+1}(\rho)+\rho_c)-(1+\rho)\delta\partial_{w_{n+1}}H_{n+1}^M,\\
&\partial_{y_n}H_n^L=-\rho_s\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}}-\rho_s(\delta\partial_{x_{n+1}}H_{n+1}^M-\gamma_{n+1}(\rho)+\rho_c).
\end{aligned}
$$

At the kink point $y_n=x_n$, we have

$$
\begin{aligned}
&\partial_{y_n}G_n\,|_{y_n\nearrow x_n}-\partial_{y_n}G_n\,|_{y_n\searrow x_n}\\
=&\partial_{y_n}G_n^U\,|_{y_n\nearrow x_n}-\partial_{y_n}G_n^M\,|_{y_n\searrow x_n}\\
=&\partial_{y_n}H_n^U+(1+\rho_c)\rho_c-\left[\partial_{y_n}H_n^M+(1+\rho_c)\gamma_n(\rho)\right]
\end{aligned}
$$

$$
\begin{aligned}
=&(\rho-\rho_c)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho)\delta\partial_{w_{n+1}}H^M_{n+1}\\
&-(1+\rho_c)(\delta\partial_{x_{n+1}}H^M_{n+1}-\gamma_{n+1}(\rho)+\rho_c)-(1+\rho_c)(\gamma_n(\rho)-\rho_c)\\
=&(\rho-\rho_c)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho)\partial_{w_{n+1}}G^M_{n+1}-(1+\rho_c)(\partial_{x_{n+1}}G^M_{n+1}+\gamma_n(\rho)-\rho_c)\\
\le&(1+\rho)\partial_{w_{n+1}}G^M_{n+1}-(1+\rho_c)(\partial_{x_{n+1}}G^M_{n+1}+\gamma_n(\rho)-\rho_c)\\
\le&(\rho-\rho_c)\partial_{w_{n+1}}G^M_{n+1}\le 0,
\end{aligned}
$$

where the first inequality follows from $\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}\le\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=M_{n+1}}=0$ (due to convexity), the second inequality follows from the induction assumption of claim (ii), and the last inequality follows from $\partial_{w_{n+1}}G^M_{n+1}\le 0$. At the kink point $y_n=x_n+w_n$, we have

$$
\begin{aligned}
&\partial_{y_n}G_n\mid_{y_n\nearrow(x_n+w_n)}-\partial_{y_n}G_n\mid_{y_n\searrow(x_n+w_n)}\\
=&\partial_{y_n}G^M_n\mid_{y_n\nearrow(x_n+w_n)}-\partial_{y_n}G^L_n\mid_{y_n\searrow(x_n+w_n)}\\
=&\partial_{y_n}H^M_n+(1+\rho_c)\gamma_n(\rho)-\left[\partial_{y_n}H^L_n+(1+\rho_c)\rho_s\right]\\
=&(\rho_s-\rho)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho_s)(\delta\partial_{x_{n+1}}H^M_{n+1}-\gamma_{n+1}(\rho)+\rho_c)\\
&-(1+\rho)\delta\partial_{w_{n+1}}H^M_{n+1}-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
=&(\rho_s-\rho)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho_s)\partial_{x_{n+1}}G^M_{n+1}-(1+\rho)\partial_{w_{n+1}}G^M_{n+1}\\
&-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
\le&(1+\rho_s)\partial_{x_{n+1}}G^M_{n+1}-(1+\rho)\partial_{w_{n+1}}G^M_{n+1}-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
\le&(1+\rho_s)\partial_{x_{n+1}}G^M_{n+1}-(1+\gamma_{N-n}(\rho))\partial_{w_{n+1}}G^M_{n+1}-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
=&(1+\gamma_{N-n}(\rho))\left(\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}}G^M_{n+1}-\partial_{w_{n+1}}G^M_{n+1}\right)-(1+\rho_c)(\rho_s-\gamma_n(\rho))\le 0,
\end{aligned}
$$

where the first inequality follows from $\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}\le\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=M_{n+1}}=0$ (due to convexity), the second inequality follows from $\gamma_{N-n}(\rho)\ge\gamma_N(\rho)=\rho$ and $\partial_{w_{n+1}}G^M_{n+1}\le 0$, and the third inequality is from the induction assumption of claim (ii). Therefore, G_n is convex in y_n at the two kink points. Similarly, when $\xi_n\ge y_n$, it can be verified that G_n is convex in y_n at the two kink points.

Second, with respect to x_n, we have

$$
\begin{aligned}
\partial_{x_n}H^U_n&=(1+\rho_c)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho_c)(\delta\partial_{x_{n+1}}H^M_{n+1}-\gamma_{n+1}(\rho)+\rho_c), \qquad \text{(A.6)}\\
\partial_{x_n}H^M_n&=(1+\rho)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho)\delta\partial_{w_{n+1}}H^M_{n+1},\\
\partial_{x_n}H^L_n&=(1+\rho_s)\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}+(1+\rho_s)(\delta\partial_{x_{n+1}}H^M_{n+1}-\gamma_{n+1}(\rho)+\rho_c).
\end{aligned}
$$

By using the induction assumption of claim (ii) and the fact that $\partial_\theta G^M_{n+1}\le 0,\theta\in\{x_{n+1},w_{n+1}\}$, and $\partial_{y_{n+1}}G^M_{n+1}\mid_{y_{n+1}=x_{n+1}+w_{n+1}}\le 0$ (discussed above), it can be verified that G_n is convex in x_n at the two kink points.

Finally, with respect to w_n, we have

$$\partial_{w_n} H_n^U = \partial_{w_n} H_n^M = (1+\rho)\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}+w_{n+1}} + (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M, \tag{A.7}$$
$$\partial_{w_n} H_n^L = (1+\rho_s)\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}+w_{n+1}} + (1+\rho_s)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c).$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_\theta G_{n+1}^M \leq 0, \theta \in \{x_{n+1}, w_{n+1}\}$, and $\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}+w_{n+1}} \leq 0$ (discussed above), it can be verified that G_n is convex in w_n at the two kink points. Therefore, we have verified that $G_n(y_n, x_n, w_n)$ is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$.

(3) For any ξ_n such that $x_{n+1} < M_{n+1} \leq x_{n+1} + w_{n+1}$: First, with respect to y_n, when $\xi_n < y_n$,

$$\partial_{y_n} H_n^U = -\rho_c(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c),$$
$$\partial_{y_n} H_n^M = (\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) - (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M,$$
$$\partial_{y_n} H_n^L = -\rho_s(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c).$$

At the kink point $y_n = x_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n \mid_{y_n \searrow x_n} \\
=&\partial_{y_n} G_n^U \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n^M \mid_{y_n \searrow x_n} \\
=&\partial_{y_n} H_n^U + (1+\rho_c)\rho_c - \left[\partial_{y_n} H_n^M + (1+\rho_c)\gamma_n(\rho)\right] \\
=&(1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M - (1+\rho_c)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) - (1+\rho_c)(\gamma_n(\rho) - \rho_c) \\
=&(1+\rho)\partial_{w_{n+1}} G_{n+1}^M - (1+\rho_c)(\partial_{x_{n+1}} G_{n+1}^M + \gamma_n(\rho) - \rho_c) \\
\leq&(\rho - \rho_c)\partial_{w_{n+1}} G_{n+1}^M \leq 0,
\end{aligned}$$

where the first inequality follows from the induction assumption of claim (ii), and the second inequality follows from $\partial_{w_{n+1}} G_{n+1}^M \leq 0$. At the kink point $y_n = x_n + w_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G_n \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} G_n^M \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G_n^L \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} H_n^M + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n} H_n^L + (1+\rho_c)\rho_s\right] \\
=&(1+\rho_s)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) - (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
=&(1+\rho_s)\partial_{x_{n+1}} G_{n+1}^M - (1+\rho)\partial_{w_{n+1}} G_{n+1}^M - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
\leq&(1+\rho_s)\partial_{x_{n+1}} G_{n+1}^M - (1+\gamma_{N-n}(\rho))\partial_{w_{n+1}} G_{n+1}^M - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
=&(1+\gamma_{N-n}(\rho))\left(\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}} G_{n+1}^M - \partial_{w_{n+1}} G_{n+1}^M\right) - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \leq 0,
\end{aligned}$$

where the first inequality follows from $\gamma_{N-n}(\rho) \geq \gamma_N(\rho) = \rho$ and $\partial_{w_{n+1}} G_{n+1}^M \leq 0$, and the second inequality follows from the induction assumption of claim (ii). Therefore, G_n is convex in y_n at the two kink points. Similarly, when $\xi_n \geq y_n$, it can be verified that G_n is convex in y_n at the two kink points.

Second, with respect to x_n, we have

$$\begin{aligned}
\partial_{x_n} H_n^U &= (1+\rho_c)(\delta \partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) = (1+\rho_c)\partial_{x_{n+1}} G_{n+1}^M, \\
\partial_{x_n} H_n^M &= (1+\rho)\delta \partial_{w_{n+1}} H_{n+1}^M, \\
\partial_{x_n} H_n^L &= (1+\rho_s)(\delta \partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c).
\end{aligned} \tag{A.8}$$

By using the induction assumption of claim (ii) and the fact that $\partial_\theta G_{n+1}^M \leq 0, \theta \in \{x_{n+1}, w_{n+1}\}$, it can be verified that G_n is convex in x_n at the two kink points.

Finally, with respect to w_n, we have

$$\begin{aligned}
\partial_{w_n} H_n^U &= \partial_{w_n} H_n^M = (1+\rho)\delta \partial_{w_{n+1}} H_{n+1}^M = (1+\rho)\partial_{w_{n+1}} G_{n+1}^M, \\
\partial_{w_n} H_n^L &= (1+\rho_s)(\delta \partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c).
\end{aligned} \tag{A.9}$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_\theta G_{n+1}^M \leq 0, \theta \in \{x_{n+1}, w_{n+1}\}$, it can be verified that G_n is convex in w_n at the two kink points. Therefore, we have verified that $G_n(y_n, x_n, w_n)$ is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$.

(4) For any ξ_n such that $M_{n+1} \leq x_{n+1} < U_{n+1}(w_{n+1})$, in this case,

$$\begin{aligned}
V_{n+1}(x_{n+1}, w_{n+1}) =& \rho_c x_{n+1} + \rho_d E\left[(\xi_{n+1} - x_{n+1})^+\right] + \delta H_{n+1}^U(x_{n+1}, x_{n+1}, w_{n+1}) \\
=& \rho_c x_{n+1} + \rho_d E\left[(\xi_{n+1} - x_{n+1})^+\right] + \delta H_{n+1}^M(x_{n+1}, x_{n+1}, w_{n+1}).
\end{aligned}$$

The last equation holds because $y_{n+1} = x_{n+1}$ is the kink point of the cash balances x_{n+2} and w_{n+2}, and thus, $H_{n+1}^U = H_{n+1}^M = E[V_{n+2}(x_{n+2}, w_{n+2})]$ at $y_{n+1} = x_{n+1}$. First, with respect to y_n, when $\xi_n < y_n$,

$$\begin{aligned}
\partial_{y_n} H_n^U &= -\rho_c \partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} - \rho_c(\delta \partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c), \\
\partial_{y_n} H_n^M &= \partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} + (\delta \partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) - (1+\rho)\delta \partial_{w_{n+1}} H_{n+1}^M \\
&= \partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} + \delta \partial_{x_{n+1}} H_{n+1}^U - (1+\rho)\delta \partial_{w_{n+1}} H_{n+1}^U, \\
\partial_{y_n} H_n^L &= -\rho_s \partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} - \rho_s \delta \partial_{x_{n+1}} H_{n+1}^U.
\end{aligned}$$

At the kink point $y_n = x_n$, we have

$$\partial_{y_n} G_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n \mid_{y_n \searrow x_n}$$

$$
\begin{aligned}
&=\partial_{y_n} G_n^U \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n^M \mid_{y_n \searrow x_n} \\
&=\partial_{y_n} H_n^U + (1+\rho_c)\rho_c - \left[\partial_{y_n} H_n^M + (1+\rho_c)\gamma_n(\rho)\right] \\
&=-(1+\rho_c)\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} + (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M \\
&\quad -(1+\rho_c)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) - (1+\rho_c)(\gamma_n(\rho) - \rho_c) \\
&=-(1+\rho_c)\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} + (1+\rho)\partial_{w_{n+1}} G_{n+1}^M - (1+\rho_c)(\partial_{x_{n+1}} G_{n+1}^M + \gamma_n(\rho) - \rho_c) \\
&\le (1+\rho)\partial_{w_{n+1}} G_{n+1}^M - (1+\rho_c)(\partial_{x_{n+1}} G_{n+1}^M + \gamma_n(\rho) - \rho_c) \\
&\le (\rho - \rho_c)\partial_{w_{n+1}} G_{n+1}^M \le 0,
\end{aligned}
$$

where the first inequality follows from $\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} \ge \partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=M_{n+1}} = 0$ (due to convexity), the second inequality follows from the induction assumption of claim (ii), and the last inequality follows from $\partial_{w_{n+1}} G_{n+1}^M \le 0$. At the kink point $y_n = x_n + w_n$, we have

$$
\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G_n \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} G_n^M \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G_n^L \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} H_n^M + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n} H_n^L + (1+\rho_c)\rho_s\right] \\
=&(1+\rho_s)\partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} + (1+\rho_s)\delta\partial_{x_{n+1}} H_{n+1}^U - (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^U - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
=&(1+\rho_s)\partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} + (1+\rho_s)\partial_{x_{n+1}} G_{n+1}^U - (1+\rho)\partial_{w_{n+1}} G_{n+1}^U - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
\le&(1+\rho_s)\partial_{x_{n+1}} G_{n+1}^U - (1+\rho)\partial_{w_{n+1}} G_{n+1}^U - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
\le&(1+\rho_s)\partial_{x_{n+1}} G_{n+1}^U - (1+\gamma_{N-n}(\rho))\partial_{w_{n+1}} G_{n+1}^U - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
=&(1+\gamma_{N-n}(\rho))\left(\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}} G_{n+1}^U - \partial_{w_{n+1}} G_{n+1}^U\right) - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \le 0,
\end{aligned}
$$

where the first inequality follows from $\partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} \le \partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=U_{n+1}(w_{n+1})} = 0$ (due to convexity), the second inequality follows from $\gamma_{N-n}(\rho) \ge \gamma_N(\rho) = \rho$ and $\partial_{w_{n+1}} G_{n+1}^U \le 0$, and the third inequality is from the induction assumption of claim (ii). Therefore, G_n is convex in y_n at the two kink points. Similarly, when $\xi_n \ge y_n$, it can be verified that G_n is convex in y_n at the two kink points.

Second, with respect to x_n, we have

$$
\begin{aligned}
&\partial_{x_n} H_n^U = (1+\rho_c)\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}} + (1+\rho_c)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) \\
\le&(1+\rho_c)(\delta\partial_{x_{n+1}} H_{n+1}^M - \gamma_{n+1}(\rho) + \rho_c) = (1+\rho_c)\partial_{x_{n+1}} G_{n+1}^M, \\
&\partial_{x_n} H_n^M = (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^M = (1+\rho)\delta\partial_{w_{n+1}} H_{n+1}^U, \\
&\partial_{x_n} H_n^L = (1+\rho_s)\partial_{y_{n+1}} G_{n+1}^U \mid_{y_{n+1}=x_{n+1}} + (1+\rho_s)\delta\partial_{x_{n+1}} H_{n+1}^U.
\end{aligned}
\tag{A.10}
$$

By using the induction assumption of claim (ii) and the fact that $\partial_\theta G^M_{n+1} \le 0, \theta \in \{x_{n+1}, w_{n+1}\}$, $\partial_{y_{n+1}} G^M_{n+1} \mid_{y_{n+1}=x_{n+1}} \ge 0$ and $\partial_{y_{n+1}} G^U_{n+1} \mid_{y_{n+1}=x_{n+1}} \le 0$ (discussed above), it can be verified that G_n is convex in x_n at the two kink points.

Finally, with respect to w_n, we have

$$\begin{aligned}
&\partial_{w_n} H^U_n = \partial_{w_n} H^M_n = (1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1} = (1+\rho)\partial_{w_{n+1}} G^U_{n+1}, \\
&\partial_{w_n} H^L_n = (1+\rho_s)\partial_{y_{n+1}} G^U_{n+1} \mid_{y_{n+1}=x_{n+1}} + (1+\rho_s)\delta\partial_{x_{n+1}} H^U_{n+1}.
\end{aligned} \tag{A.11}$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_\theta G^M_{n+1} \le 0, \theta \in \{x_{n+1}, w_{n+1}\}$, $\partial_{y_{n+1}} G^M_{n+1} \mid_{y_{n+1}=x_{n+1}} \ge 0$ and $\partial_{y_{n+1}} G^U_{n+1} \mid_{y_{n+1}=x_{n+1}} \le 0$ (discussed above), it can be verified that G_n is convex in w_n at the two kink points. Therefore, we have verified that $G_n(y_n, x_n, w_n)$ is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$.

(5) For any ξ_n such that $x_{n+1} \ge U_{n+1}(w_{n+1})$, First, with respect to y_n, when $\xi_n < y_n$,

$$\begin{aligned}
&\partial_{y_n} H^U_n = -\rho_c \delta \partial_{x_{n+1}} H^U_{n+1}, \\
&\partial_{y_n} H^M_n = \delta\partial_{x_{n+1}} H^U_{n+1} - (1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1}, \\
&\partial_{y_n} H^L_n = -\rho_s \delta \partial_{x_{n+1}} H^U_{n+1}.
\end{aligned}$$

At the kink point $y_n = x_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G_n \mid_{y_n \searrow x_n} \\
=&\partial_{y_n} G^U_n \mid_{y_n \nearrow x_n} - \partial_{y_n} G^M_n \mid_{y_n \searrow x_n} \\
=&\partial_{y_n} H^U_n + (1+\rho_c)\rho_c - \left[\partial_{y_n} H^M_n + (1+\rho_c)\gamma_n(\rho)\right] \\
=&(1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1} - (1+\rho_c)\delta\partial_{x_{n+1}} H^U_{n+1} - (1+\rho_c)(\gamma_n(\rho) - \rho_c) \\
=&(1+\rho)\partial_{w_{n+1}} G^U_{n+1} - (1+\rho_c)(\partial_{x_{n+1}} G^U_{n+1} + \gamma_n(\rho) - \rho_c) \\
\le&(\rho - \rho_c)\partial_{w_{n+1}} G^U_{n+1} \le 0,
\end{aligned}$$

where the first inequality follows from the induction assumption of claim (ii), and the second inequality follows from $\partial_{w_{n+1}} G^U_{n+1} \le 0$. At the kink point $y_n = x_n + w_n$, we have

$$\begin{aligned}
&\partial_{y_n} G_n \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G_n \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} G^M_n \mid_{y_n \nearrow (x_n+w_n)} - \partial_{y_n} G^L_n \mid_{y_n \searrow (x_n+w_n)} \\
=&\partial_{y_n} H^M_n + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n} H^L_n + (1+\rho_c)\rho_s\right] \\
=&(1+\rho_s)\delta\partial_{x_{n+1}} H^U_{n+1} - (1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1} - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
=&(1+\rho_s)\partial_{x_{n+1}} G^U_{n+1} - (1+\rho)\partial_{w_{n+1}} G^U_{n+1} - (1+\rho_c)(\rho_s - \gamma_n(\rho))
\end{aligned}$$

$$
\begin{aligned}
&\le (1+\rho_s)\partial_{x_{n+1}} G^U_{n+1} - (1+\gamma_{N-n}(\rho))\partial_{w_{n+1}} G^U_{n+1} - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
&= (1+\gamma_{N-n}(\rho))\left(\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}} G^U_{n+1} - \partial_{w_{n+1}} G^U_{n+1}\right) - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \le 0,
\end{aligned}
$$

where the first inequality follows from $\gamma_{N-n}(\rho) \ge \gamma_N(\rho) = \rho$ and $\partial_{w_{n+1}} G^U_{n+1} \le 0$, and the second inequality is from the induction assumption of claim (ii). Therefore, G_n is convex in y_n at the two kink points. Similarly, when $\xi_n \ge y_n$, it can be verified that G_n is convex in y_n at the two kink points.

Second, with respect to x_n, we have

$$
\begin{aligned}
&\partial_{x_n} H^U_n = (1+\rho_c)\delta\partial_{x_{n+1}} H^U_{n+1} = (1+\rho_c)\partial_{x_{n+1}} G^U_{n+1}, \\
&\partial_{x_n} H^M_n = (1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1}, \\
&\partial_{x_n} H^L_n = (1+\rho_s)\delta\partial_{x_{n+1}} H^U_{n+1}.
\end{aligned} \tag{A.12}
$$

By using the induction assumption of claim (ii) and the fact that $\partial_\theta G^U_{n+1} \le 0, \theta \in \{x_{n+1}, w_{n+1}\}$, it can be verified that G_n is convex in x_n at the two kink points.

Finally, with respect to w_n, we have

$$
\begin{aligned}
&\partial_{w_n} H^U_n = \partial_{w_n} H^M_n = (1+\rho)\delta\partial_{w_{n+1}} H^U_{n+1} = (1+\rho)\partial_{w_{n+1}} G^U_{n+1}, \\
&\partial_{w_n} H^L_n = (1+\rho_s)\delta\partial_{x_{n+1}} H^U_{n+1}.
\end{aligned} \tag{A.13}
$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_\theta G^U_{n+1} \le 0, \theta \in \{x_{n+1}, w_{n+1}\}$, it can be verified that G_n is convex in w_n at the two kink points. Therefore, we have verified that $G_n(y_n, x_n, w_n)$ is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$.

We have shown that given ξ_n, G_n is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$. Taking expectation over ξ_n yields that G_n is convex in y_n, x_n, and w_n at the two kink points $y_n = x_n$ and $y_n = x_n + w_n$. Together with the convexity of G^i_n for $i \in \{U, M, L\}$ in three subproblems, it follows that G_n is convex in y_n, x_n, and w_n, which verifies that claim (i) holds for period n.

Now, based on the first-order derivative of H^i_n for $i \in \{U, M, L\}$, in the five cases above, we verify claim (ii) holds for period n. We first observe from the five cases that $\partial_{x_n} H^i_n = \partial_{w_n} H^i_n, i \in \{M, L\}$. This implies that

$$
\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\partial_{x_n} G^M_n \le \partial_{x_n} G^M_n = \delta\partial_{x_n} H^M_n - \gamma_n(\rho) + \rho_c \le \delta\partial_{w_n} H^M_n = \partial_{w_n} G^M_n
$$

and

$$
\partial_{w_n} G^M_n = \delta\partial_{w_n} H^M_n = \delta\partial_{x_n} H^M_n \le (\delta\partial_{x_n} H^M_n - \gamma_n(\rho) + \rho_c) + (\gamma_{n-1}(\rho) - \rho_c) = \partial_{x_n} G^M_n + \gamma_{n-1}(\rho) - \rho_c.
$$

It can be similarly verified that $\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\partial_{x_n}G_n^L \le \partial_{w_n}G_n^L \le \partial_{x_n}G_n^L + \gamma_{n-1}(\rho) - \rho_c$. Further, from the derivations in the five cases, we can observe that $\partial_{w_n}H_n^M = \partial_{w_n}H_n^U$. This implies that

$$\begin{aligned}\partial_{w_n}G_n^U = \delta\partial_{w_n}H_n^U = \delta\partial_{w_n}H_n^M = \delta\partial_{x_n}H_n^M =&\partial_{x_n}G_n^M + \gamma_n(\rho) - \rho_c\\ &\le\partial_{x_n}G_n^U + \gamma_n(\rho) - \rho_c \le \partial_{x_n}G_n^U + \gamma_{n-1}(\rho) - \rho_c,\end{aligned}$$

where the first inequality follows from $\partial_{x_n}G_n^M\,|_{x_n\nearrow y_n} \le \partial_{x_n}G_n^U\,|_{x_n\searrow y_n}$ shown in the five cases above, and the second inequality is from the fact that $\gamma_{n-1}(\rho) > \gamma_n(\rho)$.

It remains to check $\frac{1+\rho_s}{1+\gamma_{N-n+1}}\partial_{x_n}G_n^U \le \partial_{w_n}G_n^U$ in claim (ii). First, in the cases of (1) and (3)-(5) of V_{n+1}, we find that $\partial_{x_n}H_n^U \le (1+\rho_c)\partial_{x_{n+1}}G_{n+1}^i$, and $\partial_{w_n}H_n^U = (1+\rho)\partial_{w_{n+1}}G_{n+1}^i$ for $i\in\{U,M,L\}$, see (A.4)-(A.5) and (A.8)-(A.13). Thus, for any $i\in\{U,M,L\}$,

$$\begin{aligned}\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\partial_{x_n}G_n^U - \partial_{w_n}G_n^U =&\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\delta\partial_{x_n}H_n^U - \delta\partial_{w_n}H_n^U\\ \le&\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\delta(1+\rho_c)\partial_{x_{n+1}}G_{n+1}^i - \delta(1+\rho)\partial_{w_{n+1}}G_{n+1}^i\\ =&\delta(1+\rho)\left[\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}}G_{n+1}^i - \partial_{w_{n+1}}G_{n+1}^i\right] \le 0.\end{aligned}$$

The last inequality follows from the induction assumption of claim (ii). Second, in the case of (2), replacing $\partial_{x_n}H_n^U$ and $\partial_{w_n}H_n^U$ with (A.6) and (A.7) yields

$$\begin{aligned}&\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\partial_{x_n}G_n^U - \partial_{w_n}G_n^U\\ =&\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\delta\partial_{x_n}H_n^U - \delta\partial_{w_n}H_n^U\\ =&\delta\left[\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}(1+\rho_c) - (1+\rho)\right]\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}}\\ &+\frac{1+\rho_s}{1+\gamma_{N-n+1}(\rho)}\delta(1+\rho_c)\partial_{x_{n+1}}G_{n+1}^M - \delta(1+\rho)\partial_{w_{n+1}}G_{n+1}^M\\ =&\delta(1+\rho)\left[\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)} - 1\right]\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}}\\ &+\delta(1+\rho)\left[\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}}G_{n+1}^M - \partial_{w_{n+1}}G_{n+1}^M\right]\\ \le&\delta(1+\rho)\left[\frac{1+\rho_s}{1+\gamma_{N-n}(\rho)}\partial_{x_{n+1}}G_{n+1}^M - \partial_{w_{n+1}}G_{n+1}^M\right] \le 0,\end{aligned}$$

where the first inequality is from the fact that $\rho_s > \gamma_{N-n}(\rho)$ and $\partial_{y_{n+1}}G_{n+1}^M\,|_{y_{n+1}=x_{n+1}+w_{n+1}} \le 0$ in this case (as discussed in the case (2)), and the last inequality follows from the induction assumption of claim (ii). Therefore, we have verified claim (ii) for period n.

Next, we prove the optimal cash policy in period n in claim (iii). To obtain this result, we first show that $y_n^M(x_n,w_n)$ and $y_n^L(x_n,w_n)$ depend on x_n+w_n only, while $y_n^U(x_n,w_n)$ depends on x_n and w_n, separately. Based on which we define the three critical levels that satisfy $L_n \le M_n \le U_n(w_n)$.

Recall that $G_n^i(y_n,x_n,w_n)$ for $i\in\{U,M,L\}$ is differentiable and convex in y_n as shown above. Also observe that $\lim_{y_n\to\pm\infty} G_n^i(y_n,x_n,w_n)=\infty$. Then the optimal cash level (13) can be uniquely determined by:

$$
\begin{aligned}
y_n^U(x_n,w_n) &= \min\{y_n \mid \partial_{y_n} G_n^U \ge 0\} = \min\{y_n \mid \rho_c+\rho_d F(y_n)-\rho_d+\delta\partial_{y_n} H_n^U \ge 0\},\\
y_n^M(x_n,w_n) &= \min\{y_n \mid \partial_{y_n} G_n^M \ge 0\} = \min\{y_n \mid \gamma_n(\rho)+\rho_d F(y_n)-\rho_d+\delta\partial_{y_n} H_n^M \ge 0\},\\
y_n^L(x_n,w_n) &= \min\{y_n \mid \partial_{y_n} G_n^L \ge 0\} = \min\{y_n \mid \rho_s+\rho_d F(y_n)-\rho_d+\delta\partial_{y_n} H_n^L \ge 0\}.
\end{aligned}
$$

Observe that the dependence of the optimal cash level $y_n^i(x_n,w_n)$ on x_n and w_n is from $\partial_{y_n} H_n^i$. From the first-order derivative of H_n^i in the five cases of V_{n+1} above, the term $\partial_{y_n} H_n^i$ only depends on x_{n+1} and $x_{n+1}+w_{n+1}$, through which it depends on x_n and w_n. Specifically, from the state transition equations (A.2) and (A.3), in the subproblems of $x_n<y_n\le x_n+w_n$ and $y_n>x_n+w_n$ (corresponding to $y_n^M(x_n,w_n)$ and $y_n^L(x_n,w_n)$), both x_{n+1} and $x_{n+1}+w_{n+1}$ depend only on x_n+w_n. Therefore, in these two subproblems, $\partial_{y_n} H_n^i$ for $i\in\{M,L\}$, depends on x_n+w_n only. Hence the unconstrained optimal solutions $y_n^M(x_n,w_n)$ and $y_n^L(x_n,w_n)$ are dependent on x_n+w_n only. With slight abuse of notation, we rewrite them as $y_n^M(x_n+w_n)$ and $y_n^L(x_n+w_n)$, respectively. Based on the dependence discussed above, we can define the three critical levels as

$$U_n(w_n) = \min\{x_n \mid y_n^U(x_n,w_n)=x_n\} = \min\{x_n \mid \partial_{y_n} G_n^U \mid_{y_n=x_n} \ge 0\}, \tag{A.14}$$

$$M_n = \min\{x_n+w_n \mid y_n^M(x_n+w_n)=x_n+w_n\} = \min\{x_n+w_n \mid \partial_{y_n} G_n^M \mid_{y_n=x_n+w_n} \ge 0\}, \tag{A.15}$$

$$L_n = \min\{x_n+w_n \mid y_n^L(x_n+w_n)=x_n+w_n\} = \min\{x_n+w_n \mid \partial_{y_n} G_n^L \mid_{y_n=x_n+w_n} \ge 0\}, \tag{A.16}$$

where the second equation of all three critical levels follows from the convexity of G_n^i in y_n. It is clear from the definitions that $U_n(w_n)$ depends only on w_n, and M_n and L_n do not depend on x_n and w_n. Observe that in the expression $\partial_{y_n} H_n^U$ in each of the five cases of V_{n+1}, only the terms $(-\partial_{y_{n+1}} G_{n+1}^M \mid_{y_{n+1}=x_{n+1}+w_{n+1}})$ and $(-\partial_{x_{n+1}} H_{n+1}^L)$ depend on w_n. Recall that G_{n+1}^M and H_{n+1}^L are convex in y_{n+1}, x_{n+1} and w_{n+1}, so these negative terms are decreasing in x_{n+1} and w_{n+1}. Note that x_{n+1} and w_{n+1} are both increasing in w_n. Therefore, $\partial_{y_n} H_n^U$ is decreasing in w_n. Hence, $\partial_{y_n} G_n^U$ is decreasing in w_n. From (A.14), it follows that $U_n(w_n)$ is increasing in w_n.

We next show that the three critical levels satisfy $L_n\le M_n\le U_n(w_n)$. First, consider the case $x_n+w_n=L_n$. In this case, $y_n^L(x_n+w_n)=L_n$, and $\partial_{y_n} G_n^L(y_n,x_n,w_n)\mid_{y_n=L_n}=0$. Since $y_n=x_n+w_n$ is a kink point of $G_n(y_n,x_n,w_n)$ and $G_n(y_n,x_n,w_n)$ is convex in y_n, we have $\partial_{y_n} G_n^M \mid_{y_n=L_n} \le \partial_{y_n} G_n^L \mid_{y_n=L_n}=0$. Since $\partial_{y_n} G_n^M$ is increasing in y_n due to convexity, we have $M_n\ge L_n$ based on definition (A.15). Second, consider the case $x_n=M_n$ and $w_n=0$. In this case, based on the definition of M_n given in (A.15), we have $\partial_{y_n} G_n^M \mid_{y_n=M_n}=0$. Since $y_n=x_n$ is a kink point of $G_n(y_n,x_n,w_n)$ and $G_n(y_n,x_n,w_n)$ is convex in y_n, we have $\partial_{y_n} G_n^U \mid_{y_n=M_n} \le \partial_{y_n} G_n^M \mid_{y_n=M_n}=0$. Because $\partial_{y_n} G_n^U$ is increasing in y_n due to convexity, we have $U_n(0) \geq M_n$ based on definition (A.14) (note that $w_n = 0$ in this case). As we have shown $U_n(w_n)$ is increasing in w_n, it follows that $U_n(w_n) \geq M_n$ for any $w_n \geq 0$. Therefore, we arrive at $L_n \leq M_n \leq U_n(w_n)$.

The optimal cash policy is based on the convexity of $G_n(y_n, x_n, w_n)$ in y_n and $L_n \leq M_n \leq U_n(w_n)$. According to the equations (A.14)-(A.16), when $y_n \leq x_n$, if $x_n \geq U_n(w_n)$, the optimal cash level decision $y_n^* = y_n^U(x_n, w_n)$; and when $x_n < U_n(w_n)$, the optimal cash level decision $y_n^* = x_n$ due to the constraint $y_n \leq x_n$. Similarly, in the case of $x_n < y_n \leq x_n + w_n$, when $x_n + w_n < M_n$, the optimal cash decision y_n^* is $x_n + w_n$ due to the constraint $y_n \leq x_n + w_n$; when $x_n < M_n$ and $x_n + w_n \geq M_n$, the optimal cash decision y_n^* is $y_n^M(x_n + w_n)$; when $x_n \geq M_n$, the optimal cash decision y_n^* is x_n due to the constraint $y_n > x_n$. Finally, in the case of $y_n > x_n + w_n$, when $x_n + w_n \geq L_n$, the optimal cash decision y_n^* is $x_n + w_n$, and when $x_n + w_n < L_n$, the optimal cash decision y_n^* is $y_n^L(x_n + w_n)$. Combining all these results yields the optimal policy structure given by claim (iii) for period n.

Finally, we prove the convexity and differentiability of $V_n(x_n, w_n)$ in x_n and w_n. Based on the convexity of $G_n(y_n, x_n, w_n)$ in x_n and w_n, from Theorem 5.7 of Rockafellar (1970), it follows that $V_n(x_n, w_n)$ is convex in x_n and w_n. Based on the optimal cash policy shown above, we can write the value function as

$$V_n(x_n, w_n) = \begin{cases} G_n^L(y_n^L(x_n + w_n), x_n, w_n) & \text{if } x_n + w_n < L_n, \\ G_n^M(x_n + w_n, x_n, w_n) & \text{if } L_n \leq x_n + w_n < M_n, \\ G_n^M(y_n^M(x_n + w_n), x_n, w_n) & \text{if } x_n < M_n \leq x_n + w_n, \\ G_n^U(x_n, x_n, w_n) & \text{if } M_n \leq x_n < U_n(w_n), \\ G_n^U(y_n^U(x_n, w_n), x_n, w_n) & \text{if } x_n \geq U_n(w_n). \end{cases}$$

Recall that G_n^i for $i \in \{U, M, L\}$ is differentiable in x_n and w_n. By applying the envelope theorem to each of the five segments of $V_n(x_n, w_n)$ as well as recognizing that G_n is differentiable in x_n and w_n at the four connecting points: $x_n + w_n = L_n$, $x_n + w_n = M_n$, $x_n = M_n$, and $x_n = U_n(w_n)$, it follows that $V_n(x_n, w_n)$ is differentiable in x_n and w_n, and thus claim (iv) holds for period n. This completes the induction proof. □

Proof of Proposition 3

First, we show that $\Psi = 0$ if cash demand ξ_n is deterministic. Specifically, if there is no cash flow uncertainty, i.e., ξ_n is deterministic and $\xi_n \leq 0$, then the supplier can set $y_n = 0$ in each period to achieve a minimum cost of $V_1(\cdot, \cdot) = 0$. Plugging this into the definition of Ψ, we then have $\Psi = 0$.

Next, we show Ψ is decreasing in ρ and increasing in W. According to the monotonicity property specified in Proposition 1, the cost function $V_1(x_1, w_1)$ is decreasing in w_1. Increasing ρ or decreasing W will decrease $W/(1+\rho)^N$, which results in the increase of $V_1(X, W/(1+\rho)^N)$. Since $V_1(X, 0)$ does not change according to ρ and W, we then have Ψ being decreasing in ρ and increasing in W. □

Proof of Proposition 4

First, the cash flow cost V_1 with the payment due date $N=1$ can be written as

$$V_1(x_1,w_1)=\min_{y_1}\left\{\rho_c y_1+\rho_d E(\xi_1-y_1)^+ +(\rho-\rho_c)\min\left\{(y_1-x_1)^+,w_1\right\}+(\rho_s-\rho_c)(y_1-x_1-w_1)^+\right\}.$$

As we have shown in the proof of Proposition 2, the objective function $G_N(y_N,x_N,w_N)$ is convex in y_N. This means that with $N=1$, the objective function $G_1(y_1,x_1,w_1)$ is also convex in y_1. According to the definition of y_n^i from (13), we can write their closed-form solutions as follows,

$$y_1^L=F^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right),y_1^M=F^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right),y_1^U=F^{-1}\left(\frac{\rho_d-\rho_c}{\rho_d}\right).$$

These three optimal cash levels are also equivalent to their corresponding critical levels L_1,M_1 and U_1. For ease of notation, we denote them as L,M and U, respectively.

Recall that the cash demand follows a normal distribution with its cumulative distribution function $F(\cdot)$. We then have $F^{-1}(\cdot)=\mu+\sigma\Phi^{-1}(\cdot)$, where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution. Denote the probability density function (PDF) of the standard normal distribution as $\phi(\cdot)$. The value of payables finance is $\Psi=\delta V_1(X,0)-\delta V_1(X,W/(1+\rho)^N)$. Based on the optimal cash policy, we rewrite the value functions $V_1(X,0)$ and $V_1(X,W/(1+\rho)^N)$ with $N=1$.

$$V_1(X,0)=\begin{cases}G_1(L,x_1,0) & \text{if } x_1<L,\\ G_1(x_1,x_1,0) & \text{if } L\le x_1<M,\\ G_1(U,x_1,0) & \text{if } x_1\ge U;\end{cases}$$

$$V_1(X,W/(1+\rho))=\begin{cases}G_1(L,x_1,w_1) & \text{if } x_1+w_1<L,\\ G_1(x_1+w_1,x_1,w_1) & \text{if } L\le x_1+w_1<M,\\ G_1(M,x_1,w_1) & \text{if } x_1+w_1\ge M \text{ and } x_1<M,\\ G_1(x_1,x_1,w_1) & \text{if } M\le x_1<U,\\ G_1(U,x_1,w_1) & \text{if } x_1\ge U.\end{cases}$$

We first rewrite the expectation formula as

$$\begin{aligned}E(\xi-y)^+&=\int_y^\infty \xi f(\xi)d\xi-y(1-F(y))=\int_y^\infty\left[\mu f(\xi)-\sigma^2 f'(\xi)\right]d\xi-y(1-F(y))\\&=(\mu-y)(1-F(y))+\sigma^2 f(y)=(\mu-y)\left[1-\Phi\left(\frac{y-\mu}{\sigma}\right)\right]+\sigma\phi\left(\frac{y-\mu}{\sigma}\right),\end{aligned}\tag{A.17}$$

where the last equation follows from the fact that $f(y)=\frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)$.

Next, we prove effects of ρ_s and σ case by case.

(1) When $x_1+w_1<L$, the value of payables finance can be written as $\Psi=\delta\left[G_1(L,x_1,0)-G_1(L,x_1,w_1)\right]=\frac{\rho_s-\rho}{(1+\rho)(1+\rho_c)}W$. Thus, Ψ is increasing in ρ_s and independent of σ.

(2) When $L \le x_1 + w_1 < M$ and $x_1 < L$, we have

$$\begin{aligned}
\delta^{-1}\Psi &= G_1(L, x_1, 0) - G_1(x_1 + w_1, x_1, w_1) \\
&= \rho_d E(\xi_1 - L)^+ - \rho_d E(\xi_1 - x_1 - w_1)^+ + \rho_s(L - x_1) - \rho w_1 \\
&= \rho_d \int_L^\infty \xi f(\xi) d\xi - \rho_s L - \rho_d E(\xi_1 - x_1 - w_1)^+ + \rho_s(L - x_1) - \rho w_1 \\
&= \mu\rho_s + \sigma\rho_d \phi\left(\Phi^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)\right) - \rho_d(\mu - x_1 - w_1)\left[1 - \Phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)\right] \\
&\quad - \sigma\rho_d\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right) - \rho_s x_1 - \rho w_1.
\end{aligned}$$

The last equation follows from (A.17). Taking the derivative of the third equation above with respect to ρ_s yields

$$\partial_{\rho_s}\delta^{-1}\Psi = \rho_d\left(-Lf(L)\right)\partial_{\rho_s}L - x_1 = L - x_1 > 0,$$

where $f(L) = \frac{1}{\sigma}\phi\left(\Phi^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)\right)$ and $\partial_{\rho_s}L = -\sigma\frac{1}{\rho_d}\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)' = -\sigma\frac{1}{\rho_d}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)}$. Thus, Ψ is increasing in ρ_s. Similarly, taking the derivative of the last equation of $\delta^{-1}\Psi$ above with respect to σ yields

$$\begin{aligned}
\partial_\sigma\delta^{-1}\Psi =& \rho_d\phi\left(\Phi^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)\right) + \rho_d(\mu - x_1 - w_1)^2\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)\sigma^{-2} \\
&- \rho_d\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right) - \sigma\rho_d\left(-\frac{(x_1 + w_1 - \mu)^2}{\sigma}\right)\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)(-\sigma^{-2}) \\
=& \rho_d\phi\left(\Phi^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right)\right) - \rho_d\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right).
\end{aligned}$$

Suppose that $\rho_d > 2\rho_s$. Then, we have $\frac{\rho_d-\rho_s}{\rho_d} \ge \frac{1}{2}$, and thus $\frac{x_1+w_1-\mu}{\sigma} \ge \frac{L-\mu}{\sigma} = \Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right) \ge 0$. This yields $\partial_\sigma\delta^{-1}\Psi \ge 0$. Therefore, Ψ is increasing in σ when $\rho_d > 2\rho_s$.

(3) When $L \le x_1 + w_1 < M$ and $L \le x_1 < U$, we have

$$\begin{aligned}
\delta^{-1}\Psi =& G_1(x_1, x_1, 0) - G_1(x_1 + w_1, x_1, w_1) = \rho_d E(\xi_1 - x_1)^+ - \rho_d E(\xi_1 - x_1 - w_1)^+ - \rho w_1 \\
=& \rho_d(\mu - x_1)\left[1 - \Phi\left(\frac{x_1 - \mu}{\sigma}\right)\right] + \rho_d\sigma\phi\left(\frac{x_1 - \mu}{\sigma}\right) \\
&- \rho_d(\mu - x_1 - w_1)\left[1 - \Phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)\right] - \rho_d\sigma\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right) - \rho w_1.
\end{aligned}$$

In this case, Ψ is independent of ρ_s. Taking derivative with respect to σ yields

$$\begin{aligned}
\partial_\sigma\delta^{-1}\Psi =& -\rho_d(\mu - x_1)^2\phi\left(\frac{x_1 - \mu}{\sigma}\right)\sigma^{-2} + \rho_d\phi\left(\frac{x_1 - \mu}{\sigma}\right) + \rho_d\sigma\phi\left(\frac{x_1 - \mu}{\sigma}\right)\left(-\frac{(x_1 - \mu)^2}{\sigma}\right)(-\sigma^2) \\
&+ \rho_d(\mu - x_1 - w_1)^2\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)\sigma^{-2} - \rho_d\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right) \\
&- \rho_d\sigma\phi\left(\frac{x_1 + w_1 - \mu}{\sigma}\right)\left(-\frac{(x_1 + w_1 - \mu)^2}{\sigma}\right)(-\sigma^2)
\end{aligned}$$

$$=\rho_d\phi\left(\frac{x_1-\mu}{\sigma}\right)-\rho_d\phi\left(\frac{x_1+w_1-\mu}{\sigma}\right)\geq 0,$$

where the last inequality follows from $\mu\leq 0$ and $x_1\leq x_1+w_1$. Thus, Ψ is increasing in σ.

(4) When $x_1+w_1\geq M, x_1<M$ and $x_1<L$, we have

$$\begin{aligned}\delta^{-1}\Psi=&G_1(L,x_1,0)-G_1(M,x_1,w_1)=\rho_d E(\xi_1-L)^+-\rho_d E(\xi_1-M)^++\rho_s L-\rho M-(\rho_s-\rho)x_1\\ =&\sigma\rho_d\left[\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)-\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)\right)\right]+\mu(\rho_s-\rho)-(\rho_s-\rho)x_1.\end{aligned}$$

Taking its derivative with respect to ρ_s, we have

$$\begin{aligned}\partial_{\rho_s}\delta^{-1}\Psi=&\sigma\rho_d\left(-\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)\frac{1}{\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)}\left(-\frac{1}{\rho_d}\right)+\mu-x_1\\ =&\mu+\sigma\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)-x_1=L-x_1>0\end{aligned}$$

Thus, Ψ is increasing in ρ_s. We then take derivative with respect to σ, and obtain

$$\partial_\sigma\delta^{-1}\Psi=\rho_d\left[\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\right)-\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)\right)\right].$$

Suppose that $\rho_d>2\rho_s$. Then, we have $\frac{\rho_d-\rho_s}{\rho_d}\geq\frac{1}{2}$, and thus $\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)=\frac{M-\mu}{\sigma}\geq\frac{L-\mu}{\sigma}=\Phi^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)\geq 0$. This yields $\partial_\sigma\delta^{-1}\Psi\geq 0$. Therefore, Ψ is increasing in σ when $\rho_d>2\rho_s$.

(5) When $x_1+w_1\geq M, x_1<M$ and $x_1<L$, we have $L\leq x_1<U$, we have

$$\begin{aligned}\delta^{-1}\Psi=&G_1(x_1,x_1,0)-G_1(M,x_1,w_1)=\rho_d E(\xi_1-x_1)^+-\rho_d E(\xi_1-M)^+-\rho(M-x_1)\\ =&\rho_d(\mu-x_1)\left[1-\Phi\left(\frac{x_1-\mu}{\sigma}\right)\right]+\rho_d\sigma\phi\left(\frac{x_1-\mu}{\sigma}\right)-\mu\rho-\sigma\rho_d\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)\right)-\rho x_1.\end{aligned}$$

In this case, Ψ is independent of ρ_s. Taking derivative with respect to σ yields

$$\partial_\sigma\delta^{-1}\Psi=\rho_d\phi\left(\frac{x_1-\mu}{\sigma}\right)-\rho_d\phi\left(\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)\right).$$

Suppose that $\rho_d>2\rho_s$. Then, we have $\frac{\rho_d-\rho}{\rho_d}>\frac{\rho_d-\rho_s}{\rho_d}\geq\frac{1}{2}$, and thus $\Phi^{-1}\left(\frac{\rho_d-\rho}{\rho_d}\right)=\frac{M-\mu}{\sigma}\geq\frac{x_1-\mu}{\sigma}\geq 0$. This yields $\partial_\sigma\delta^{-1}\Psi\geq 0$. Therefore, Ψ is increasing in σ when $\rho_d>2\rho_s$.

(6) When $M\leq x_1<U$, the value of payables finance is $\Psi=\delta\left[G_1(x_1,x_1,0)-G_1(x_1,x_1,w_1)\right]=0$, which is independent of ρ_s and σ.

(7) When $x_1\geq U$, the value of payables finance is $\Psi=\delta\left[G_1(U,x_1,0)-G_1(U,x_1,w_1)\right]=0$, which is independent of ρ_s and σ.

Therefore, the value of payables finance Ψ is (weakly) increasing in ρ_s in all cases, and (weakly) increasing in σ when $\rho_d>2\rho_s$. □

Proof of Proposition 5

First, we show that Δ^* is decreasing in ρ. Recall from (17) that

$$\Psi(\Delta) = \delta\left[V_1(X,0) - V_1\left(X, W/(1+\rho)^{N+\Delta}\right)\right] - \frac{(1+\rho)^\Delta - 1}{(1+\rho)^\Delta}\delta^N W.$$

The first term $V_1(X,0)$ is independent of ρ since no payables finance amount exists. Regarding the second term $V_1(X, W/(1+\rho)^{N+\Delta})$, as ρ increases, the initial payables finance amount $W/(1+\rho)^{N+\Delta}$ decreases, and as a result, $V_1(X, W/(1+\rho)^{N+\Delta})$ increases according to the monotonicity property of $V_1(x_1, w_1)$ with respect to w_1 in Proposition 1. For the third term, it is clear that $\frac{(1+\rho)^\Delta - 1}{(1+\rho)^\Delta}\delta^N W$ increases in ρ. Combining all of them yields that $\Psi(\Delta)$ decreases in ρ. With the same reasoning, as Δ increases, the terms $V_1(X, W/(1+\rho)^{N+\Delta})$ and $\frac{(1+\rho)^\Delta - 1}{(1+\rho)^\Delta}\delta^N W$ also increase. Thus, $\Psi(\Delta)$ decreases in Δ. Since the equilibrium payment term extension Δ^* is defined by $\Delta^* = \max\{\Delta \geq 0 \mid \text{s.t. } \Psi(\Delta) \geq 0\}$, we have Δ^* being decreasing in ρ.

In the following, we consider the case in which $N = 1$ and cash flow follows a normal distribution. We first show that Δ^* is increasing in σ when $\rho_d > 2\rho_s$. Recall that $\Psi = \delta V_1(X,0) - \delta V_1(X, W/(1+\rho)^N)$ is increasing in σ when $\rho_d > 2\rho_s$ in Proposition 4. Since $\Psi(\Delta)$ is similar to Ψ with its different terms being independent of σ, then $\Psi(\Delta)$ is also increasing in σ when $\rho_d > 2\rho_s$. Because $\Psi(\Delta)$ is decreasing in Δ as shown above, we can obtain that Δ^* is increasing in σ when $\rho_d > 2\rho_s$.

We then show that Δ^* may be either increasing or decreasing in W, by providing two cases discussed in the proof of Proposition 4. In the first case of $x_1 + w_1 < L$, we have

$$\Psi(\Delta) = \frac{\rho_s - \rho}{(1+\rho)^{\Delta+1}(1+\rho_c)}W - \frac{(1+\rho)^\Delta - 1}{(1+\rho)^\Delta}\delta^N W = \frac{1}{(1+\rho)^{\Delta+1}(1+\rho_c)}W\left[1 + \rho_s - (1+\rho)^{\Delta+1}\right] \geq 0.$$

The last inequality is from the fact that it is optimal for the supplier to draw from payables finance before resorting to additional short-term loans, i.e., $1 + \rho_s - (1+\rho)^{\Delta+1} \geq 0$. Thus, $\Psi(\Delta)$ is increasing in W. Because $\Psi(\Delta)$ is decreasing in Δ, we can obtain that in this case, Δ^* is increasing in W. However, in the second case of $x_1 \geq M$, we have $\Psi(\Delta) = -\frac{(1+\rho)^\Delta - 1}{(1+\rho)^\Delta}\delta^N W$, which is decreasing in W. Opposite to the first case, we have that Δ^* is decreasing in W. As a result, Δ^* may be either increasing or decreasing in W, depending on the level of cash flow uncertainty. □

Proof of Proposition 6

We first prove the system cost lower bound $\tilde{V}_n(x_n, w_n) \leq V_n(x_n, w_n)$ for any $1 \leq n \leq N$ using backward (and forward) induction and the monotonicity property of $V_n(x_n, w_n)$ in Proposition 1. We then use backward induction again to show the convexity of $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ and characterize the optimal cash policy in the approximate system. Its proof is similar to the proof of Proposition 2.

We finally establish the properties of middle cash level in the optimal cash policy using backward induction.

System cost lower bound. To show $\tilde{V}_n(x_n, w_n) \leq V_n(x_n, w_n)$ for any $1 \leq n \leq N$, we first use forward induction to prove $x_n \leq \tilde{x}_n$ and $w_n \leq \tilde{w}_n$ for any $1 \leq n \leq N$. In the first period, we have $\tilde{x}_1 = x_1$ and $\tilde{w}_1 = w_1$. Suppose that $x_n \leq \tilde{x}_n$ and $w_n \leq \tilde{w}_n$ for any $1 \leq n < N$. Using forward induction, we need to prove $x_{n+1} \leq \tilde{x}_{n+1}$ and $w_{n+1} \leq \tilde{w}_{n+1}$. According to the state transitions, for $1 \leq n < N$,

$$\begin{aligned} x_{n+1} =& (1+\rho)\left[x_n + \min\{(y_n - x_n)^+, w_n\} - \xi_n\right] \\ & - (\rho - \rho_c)(x_n - y_n)^+ - (\rho_s - \rho)(y_n - x_n - w_n)^+ - \rho(y_n - \xi_n)^+ - (\rho_d - \rho)(\xi_n - y_n)^+ \\ \leq& (1+\rho)\left[x_n + \min\{(y_n - x_n)^+, w_n\} - \xi_n\right] \\ \leq& (1+\rho)\left[\tilde{x}_n + \min\{(y_n - \tilde{x}_n)^+, \tilde{w}_n\} - \xi_n\right] = \tilde{x}_{n+1}, \end{aligned}$$

where the second inequality is from the fact that $[x_n + \min\{(y_n - x_n)^+, w_n\}]$ is increasing in x_n and w_n. Thus, we have $x_n \leq \tilde{x}_n$ for any $1 \leq n \leq N$. Similarly, for $1 \leq n < N$,

$$w_{n+1} = (1+\rho)\left[w_n - (y_n - x_n)^+\right]^+ \leq (1+\rho)\left[\tilde{w}_n - (y_n - \tilde{x}_n)^+\right]^+ = \tilde{w}_{n+1},$$

where the inequality follows from the fact that $[w_n - (y_n - x_n)^+]^+$ is increasing in x_n and w_n. Thus, we have $w_n \leq \tilde{w}_n$ for any $1 \leq n \leq N$. Based on this result and the monotonicity property of the value function $V_n(x_n, w_n)$ shown in Proposition 1, we next show $V_n(x_n, w_n) \geq \tilde{V}_n(x_n, w_n)$ for any $1 \leq n \leq N$ using backward induction. Starting from the last period N, since $V_{N+1}(\cdot,\cdot) = \tilde{V}_{N+1}(\cdot,\cdot) = 0$, we have $V_N(x_N, w_N) = \tilde{V}_N(x_N, w_N)$. Suppose that $V_{n+1}(x_{n+1}, w_{n+1}) \geq \tilde{V}_{n+1}(x_{n+1}, w_{n+1})$ for any $1 \leq n < N-1$. We check the following:

$$\begin{aligned} V_n(x_n, w_n) =& \min_{y_n}\Big\{\rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)\min\big\{(y_n - x_n)^+, w_n\big\} \\ & + (\rho_s - \rho_c)(y_n - x_n - w_n)^+ + \delta E\left[V_{n+1}(x_{n+1}, w_{n+1})\right]\Big\} \\ \geq& \min_{y_n}\Big\{\rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)\min\big\{(y_n - x_n)^+, w_n\big\} \\ & + (\rho_s - \rho_c)(y_n - x_n - w_n)^+ + \delta E\left[V_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1})\right]\Big\} \\ \geq& \min_{y_n}\Big\{\rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)\min\big\{(y_n - x_n)^+, w_n\big\} \\ & + (\rho_s - \rho_c)(y_n - x_n - w_n)^+ + \delta E\left[\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1})\right]\Big\} = \tilde{V}_n(x_n, w_n), \end{aligned}$$

where the first inequality follows from $x_{n+1} \leq \tilde{x}_{n+1}$, $w_{n+1} \leq \tilde{w}_{n+1}$ and the monotonicity of V_{n+1} in Proposition 1, and the second inequality follows from $V_{n+1}(x_{n+1}, w_{n+1}) \geq \tilde{V}_{n+1}(x_{n+1}, w_{n+1})$ for any x_{n+1} and w_{n+1} in the backward induction assumption. Therefore, we have $V_n(x_n, w_n) \geq \tilde{V}_n(x_n, w_n)$ for any $1 \leq n \leq N$.

Convexity of $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ and the optimal cash policy. The proof is similar to the proof of Proposition 2. The state transition equations (20) and (21) in the approximate system can be written as:

$$\tilde{x}_{n+1} = \begin{cases} (1+\rho)(\tilde{x}_n - \xi_n) & \text{if } y_n \le \tilde{x}_n, \\ (1+\rho)(y_n - \xi_n) & \text{if } \tilde{x}_n < y_n \le \tilde{x}_n + \tilde{w}_n, \\ (1+\rho)(\tilde{x}_n + \tilde{w}_n - \xi_n) & \text{if } y_n > \tilde{x}_n + \tilde{w}_n; \end{cases} \tag{A.18}$$

$$\tilde{w}_{n+1} = \begin{cases} (1+\rho)\tilde{w}_n & \text{if } y_n \le \tilde{x}_n, \\ (1+\rho)(\tilde{w}_n + \tilde{x}_n - y_n) & \text{if } \tilde{x}_n < y_n \le \tilde{x}_n + \tilde{w}_n, \\ 0 & \text{if } y_n > \tilde{x}_n + \tilde{w}_n. \end{cases} \tag{A.19}$$

Similar to (7)-(9), we define the three subproblem objective functions as $\tilde{G}_n^U(y_n, \tilde{x}_n, \tilde{w}_n)$, $\tilde{G}_n^M(y_n, \tilde{x}_n, \tilde{w}_n)$ and $\tilde{G}_n^L(y_n, \tilde{x}_n, \tilde{w}_n)$, and the three expected cost-to-go functions as $\tilde{H}_n^U(y_n, \tilde{x}_n, \tilde{w}_n)$, $\tilde{H}_n^M(y_n, \tilde{x}_n, \tilde{w}_n)$ and $\tilde{H}_n^L(y_n, \tilde{x}_n, \tilde{w}_n)$ according to the three decision cases $y_n \le \tilde{x}_n$, $\tilde{x}_n < y_n \le \tilde{x}_n + \tilde{w}_n$ and $y_n > \tilde{x}_n + \tilde{w}_n$.

Again, for ease of notation, if no confusion arises, we drop the arguments y_n, $\tilde{x}_n$, and $\tilde{w}_n$ in $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$, $\tilde{G}_n^i(y_n, \tilde{x}_n, \tilde{w}_n)$, and $\tilde{H}_n^i(y_n, \tilde{x}_n, \tilde{w}_n)$ for $i \in \{U, M, L\}$. In what follows, we use the backward induction to prove the following claims together: for $1 \le n \le N$,

(i) $\tilde{G}_n^i$, $i \in \{U, M, L\}$, is differentiable in y_n, $\tilde{x}_n$, and $\tilde{w}_n$. G_n is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$;

(ii) $\rho_c - \gamma_n(\rho) \le \partial_{\tilde{x}_n}\tilde{G}_n^U - \partial_{\tilde{w}_n}\tilde{G}_n^U \le 0$, and $\partial_{\tilde{x}_n}\tilde{G}_n^i - \partial_{\tilde{w}_n}\tilde{G}_n^i = \rho_c - \gamma_n(\rho)$ for $i \in \{M, L\}$; Moreover, $\partial_{\tilde{x}_n}\tilde{G}_n^U - \partial_{\tilde{w}_n}\tilde{G}_n^U$ only depends on $\tilde{x}_n$ (not on $\tilde{w}_n$);

(iii) There exist three optimal cash levels $L \le \tilde{M}_n \le U$, such that the optimal cash policy $y_n^\dagger$ is given by

$$y_n^\dagger = \begin{cases} L = F^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right) & \text{if } \tilde{x}_n + \tilde{w}_n < L, \\ \tilde{x}_n + \tilde{w}_n & \text{if } L \le \tilde{x}_n + \tilde{w}_n < \tilde{M}_n, \\ \tilde{M}_n & \text{if } \tilde{x}_n < \tilde{M}_n \le \tilde{x}_n + \tilde{w}_n, \\ \tilde{x}_n & \text{if } \tilde{M}_n \le \tilde{x}_n < U, \\ U = F^{-1}(\frac{\rho_d - \rho_c}{\rho_d}) & \text{if } \tilde{x}_n \ge U; \end{cases}$$

(iv) $\tilde{V}_n(\tilde{x}_n, \tilde{w}_n)$ is convex and differentiable in $\tilde{x}_n$ and $\tilde{w}_n$.

We first verify that these claims hold for the last period N. We then assume they hold for period $n+1$ and prove that they are also true for period n, completing the backward induction.

First, for period N, the dynamic program (22) in the approximate system is the same as the original dynamic program (5), due to $\tilde{V}_{N+1}(\cdot,\cdot) = V_{N+1}(\cdot,\cdot) = 0$. Thus, similar to the verification of the claims for period N in the proof of Proposition 2, it can also be similarly verified that the claims above hold for the last period N in the approximate system.

We now assume the claims hold for period $n+1$ and verify them for period n. Again, it can be similarly verified the differentiability and convexity of $\tilde{G}_n^i, i \in \{U, M, L\}$, in y_n, $\tilde{x}_n$ and $\tilde{w}_n$ (see the proof of Proposition 2). It remains to check the convexity of $\tilde{G}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$ in claim (i). From the induction assumption of claim (iii), the five exhaustive cases for the value function $\tilde{V}_n(\tilde{x}_n, \tilde{w}_n)$ can be written as the follows.

(1) if $\tilde{x}_{n+1} + \tilde{w}_{n+1} < L$,

$$\begin{aligned}\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) =& \rho_s L - (\rho_s - \rho_c)\tilde{x}_{n+1} - (\rho_s - \gamma_{n+1}(\rho))\tilde{w}_{n+1} \\ &+ \rho_d E\left[(\xi_{n+1} - L)^+\right] + \delta \tilde{H}_{n+1}^L(L, \tilde{x}_{n+1}, \tilde{w}_{n+1});\end{aligned}$$

(2) if $L \le \tilde{x}_{n+1} + \tilde{w}_{n+1} < \tilde{M}_{n+1}$,

$$\begin{aligned}\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) =& \rho_c \tilde{x}_{n+1} + \gamma_{n+1}(\rho)\tilde{w}_{n+1} + \rho_d E\left[(\xi_{n+1} - \tilde{x}_{n+1} - \tilde{w}_{n+1})^+\right] \\ &+ \delta \tilde{H}_{n+1}^M(\tilde{x}_{n+1} + \tilde{w}_{n+1}, \tilde{x}_{n+1}, \tilde{w}_{n+1});\end{aligned}$$

(3) if $\tilde{x}_{n+1} < \tilde{M}_{n+1} \le \tilde{x}_{n+1} + \tilde{w}_{n+1}$,

$$\begin{aligned}\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) =& \rho_c \tilde{M}_{n+1} + (\gamma_{n+1}(\rho) - \rho_c)(\tilde{M}_{n+1} - \tilde{x}_{n+1}) \\ &+ \rho_d E\left[(\xi_{n+1} - \tilde{M}_{n+1})^+\right] + \delta \tilde{H}_{n+1}^M(\tilde{M}_{n+1}, \tilde{x}_{n+1}, \tilde{w}_{n+1});\end{aligned}$$

(4) if $\tilde{M}_{n+1} \le \tilde{x}_{n+1} < U$,

$$\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) = \rho_c \tilde{x}_{n+1} + \rho_d E\left[(\xi_{n+1} - \tilde{x}_{n+1})^+\right] + \delta \tilde{H}_{n+1}^U(\tilde{x}_{n+1}, \tilde{x}_{n+1}, \tilde{w}_{n+1});$$

(5) if $\tilde{x}_{n+1} \ge U$,

$$\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) = \rho_c U + \rho_d E\left[(\xi_{n+1} - U)^+\right] + \delta \tilde{H}_{n+1}^U(U, \tilde{x}_{n+1}, \tilde{w}_{n+1}).$$

Note that, the optimal cash levels $L, \tilde{M}_{n+1}$ and U are independent of the cash balances $\tilde{x}_{n+1}$ and $\tilde{w}_{n+1}$. Similar to the proof of convexity at the two kink points in Proposition 2, based on these five exhaustive cases of $\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1})$ above, we obtain the following:

(1) For any ξ_n such that $\tilde{x}_{n+1} + \tilde{w}_{n+1} < L$: First, with respect to y_n, we have

$$\partial_{y_n} \tilde{H}_n^U = \partial_{y_n} \tilde{H}_n^L = 0, \quad \partial_{y_n} \tilde{H}_n^M = (1+\rho)\partial_{\tilde{x}_{n+1}} \tilde{G}_{n+1}^L - (1+\rho)\partial_{\tilde{w}_{n+1}} \tilde{G}_{n+1}^L.$$

At the kink point $y_n = \tilde{x}_n$, we have

$$\begin{aligned}&\partial_{y_n} \tilde{G}_n \,|_{y_n \nearrow \tilde{x}_n} - \partial_{y_n} \tilde{G}_n \,|_{y_n \searrow \tilde{x}_n} \\ =& \partial_{y_n} \tilde{G}_n^U \,|_{y_n \nearrow \tilde{x}_n} - \partial_{y_n} \tilde{G}_n^M \,|_{y_n \searrow \tilde{x}_n}\end{aligned}$$

$$
\begin{aligned}
&=\partial_{y_n}\tilde{H}_n^U+(1+\rho_c)\rho_c-\left[\partial_{y_n}\tilde{H}_n^M+(1+\rho_c)\gamma_n(\rho)\right]\\
&=(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L-(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L-(1+\rho_c)(\gamma_n(\rho)-\rho_c)\\
&=(1+\rho)(\gamma_{n+1}(\rho)-\rho_c)-(1+\rho_c)(\gamma_n(\rho)-\rho_c)\\
&=(1+\rho_c)\left[\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1)-(1+\rho)-\gamma_n(\rho)+\rho_c\right]\\
&=(1+\rho_c)(\rho_c-\rho)\le 0,
\end{aligned}
$$

where the last equation follows from the fact that $\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1)=\gamma_n(\rho)+1$. At the kink point $y_n=\tilde{x}_n+\tilde{w}_n$, we have

$$
\begin{aligned}
&\partial_{y_n}\tilde{G}_n\,|_{y_n\nearrow(\tilde{x}_n+\tilde{w}_n)}-\partial_{y_n}\tilde{G}_n\,|_{y_n\searrow(\tilde{x}_n+\tilde{w}_n)}\\
=&\partial_{y_n}\tilde{G}_n^M\,|_{y_n\nearrow(\tilde{x}_n+\tilde{w}_n)}-\partial_{y_n}\tilde{G}_n^L\,|_{y_n\searrow(\tilde{x}_n+\tilde{w}_n)}\\
=&\partial_{y_n}\tilde{H}_n^M+(1+\rho_c)\gamma_n(\rho)-\left[\partial_{y_n}\tilde{H}_n^L+(1+\rho_c)\rho_s\right]\\
=&(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L-(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L-(1+\rho_c)(\rho_s-\gamma_n(\rho))\\
=&(1+\rho)(\rho_c-\gamma_{n+1}(\rho))-(1+\rho_c)(\rho_s-\gamma_n(\rho))\le 0,
\end{aligned}
$$

where the last equation follows from $\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L=\rho_c-\gamma_{n+1}(\rho)$ in the induction assumption of claim (ii). Therefore, $\tilde{G}_n$ is convex in y_n at the two kink points.

Second, with respect to $\tilde{x}_n$, we have

$$\partial_{\tilde{x}_n}\tilde{H}_n^U=\partial_{\tilde{x}_n}\tilde{H}_n^L=(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L,\quad \partial_{\tilde{x}_n}\tilde{H}_n^M=(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L.$$

By using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{x}_n$ at the two kink points.

Finally, with respect to $\tilde{w}_n$, we have

$$\partial_{\tilde{w}_n}\tilde{H}_n^U=\partial_{\tilde{w}_n}\tilde{H}_n^M=(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L,\quad \partial_{\tilde{w}_n}\tilde{H}_n^L=(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L.$$

Again, by using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{w}_n$ at the two kink points. Therefore, we have verified that $\tilde{G}_n(y_n,\tilde{x}_n,\tilde{w}_n)$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n=\tilde{x}_n$ and $y_n=\tilde{x}_n+\tilde{w}_n$.

(2) For any ξ_n such that $L\le\tilde{x}_{n+1}+\tilde{w}_{n+1}<\tilde{M}_{n+1}$: First, with respect to y_n, we have

$$\partial_{y_n}\tilde{H}_n^U=\partial_{y_n}\tilde{H}_n^L=0,\quad \partial_{y_n}\tilde{H}_n^M=(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M-(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M.$$

Following from $\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M=\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^L-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^L$ in the induction assumption of claim (ii), this case is the same as the case (1). Thus, by using the induction assumption of claim (ii), it can be similarly verified that $\tilde{G}_n$ is convex in y_n at the two kink points.

Second, with respect to $\tilde{x}_n$, we have

$$\partial_{\tilde{x}_n}\tilde{H}_n^U = \partial_{\tilde{x}_n}\tilde{H}_n^L = (1+\rho)\partial_{y_{n+1}}\tilde{G}_{n+1}^M\,|_{y_{n+1}=\tilde{x}_{n+1}+\tilde{w}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M,$$
$$\partial_{\tilde{x}_n}\tilde{H}_n^M = (1+\rho)\partial_{y_{n+1}}\tilde{G}_{n+1}^M\,|_{y_{n+1}=\tilde{x}_{n+1}+\tilde{w}_{n+1}} + (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M.$$

By using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{x}_n$ at the two kink points.

Finally, with respect to $\tilde{w}_n$, we have

$$\partial_{\tilde{w}_n}\tilde{H}_n^U = \partial_{\tilde{w}_n}\tilde{H}_n^M = (1+\rho)\partial_{y_{n+1}}\tilde{G}_{n+1}^M\,|_{y_{n+1}=\tilde{x}_{n+1}+\tilde{w}_{n+1}} + (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M,$$
$$\partial_{\tilde{w}_n}\tilde{H}_n^L = (1+\rho)\partial_{y_{n+1}}\tilde{G}_{n+1}^M\,|_{y_{n+1}=\tilde{x}_{n+1}+\tilde{w}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M.$$

Again, by using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{w}_n$ at the two kink points. Therefore, we have verified that $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$.

(3) For any ξ_n such that $\tilde{x}_{n+1} < \tilde{M}_{n+1} \le \tilde{x}_{n+1} + \tilde{w}_{n+1}$: First, with respect to y_n, we have

$$\partial_{y_n}\tilde{H}_n^U = \partial_{y_n}\tilde{H}_n^L = 0, \quad \partial_{y_n}\tilde{H}_n^M = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M.$$

This is the same as the case (2) above. Thus, by using the induction assumption of claim (ii), it can be similarly verified that $\tilde{G}_n$ is convex in y_n at the two kink points.

Second, with respect to $\tilde{x}_n$, we have

$$\partial_{\tilde{x}_n}\tilde{H}_n^U = \partial_{\tilde{x}_n}\tilde{H}_n^L = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M, \quad \partial_{\tilde{x}_n}\tilde{H}_n^M = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M.$$

By using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{x}_n$ at the two kink points.

Finally, with respect to $\tilde{w}_n$, we have

$$\partial_{\tilde{w}_n}\tilde{H}_n^U = \partial_{\tilde{w}_n}\tilde{H}_n^M = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M, \quad \partial_{\tilde{w}_n}\tilde{H}_n^L = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M.$$

Again, by using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{w}_n$ at the two kink points. Therefore, we have verified that $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$.

(4) For any ξ_n such that $\tilde{M}_{n+1} \le \tilde{x}_{n+1} < U$, in this case,

$$\begin{aligned}\tilde{V}_{n+1}(\tilde{x}_{n+1}, \tilde{w}_{n+1}) =& \rho_c\tilde{x}_{n+1} + \rho_d E\left[(\xi_{n+1} - \tilde{x}_{n+1})^+\right] + \delta\tilde{H}_{n+1}^U(\tilde{x}_{n+1}, \tilde{x}_{n+1}, \tilde{w}_{n+1})\\ =& \rho_c\tilde{x}_{n+1} + \rho_d E\left[(\xi_{n+1} - \tilde{x}_{n+1})^+\right] + \delta\tilde{H}_{n+1}^M(\tilde{x}_{n+1}, \tilde{x}_{n+1}, \tilde{w}_{n+1}).\end{aligned}$$

The last equation holds because $y_{n+1} = \tilde{x}_{n+1}$ is the kink point of the cash balances $\tilde{x}_{n+2}$ and $\tilde{w}_{n+2}$, and thus, $\tilde{H}^U_{n+1} = \tilde{H}^M_{n+1} = E[\tilde{V}_{n+2}(\tilde{x}_{n+2}, \tilde{w}_{n+2})]$ at $y_{n+1} = \tilde{x}_{n+1}$. First, with respect to y_n, we have

$$\begin{aligned}\partial_{y_n}\tilde{H}^U_n =&\partial_{y_n}\tilde{H}^L_n = 0,\\ \partial_{y_n}\tilde{H}^M_n =&(1+\rho)\partial_{y_{n+1}}\tilde{G}^U_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}\\ =&(1+\rho)\partial_{y_{n+1}}\tilde{G}^M_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1} - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1}.\end{aligned}$$

At the kink point $y_n = \tilde{x}_n$, we have

$$\begin{aligned}&\partial_{y_n}\tilde{G}_n\,|_{y_n\nearrow\tilde{x}_n} - \partial_{y_n}\tilde{G}_n\,|_{y_n\searrow\tilde{x}_n}\\ =&\partial_{y_n}\tilde{G}^U_n\,|_{y_n\nearrow\tilde{x}_n} - \partial_{y_n}\tilde{G}^M_n\,|_{y_n\searrow\tilde{x}_n}\\ =&\partial_{y_n}\tilde{H}^U_n + (1+\rho_c)\rho_c - \left[\partial_{y_n}\tilde{H}^M_n + (1+\rho_c)\gamma_n(\rho)\right]\\ =&-(1+\rho)\partial_{y_{n+1}}\tilde{G}^M_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1} - (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1} - (1+\rho_c)(\gamma_n(\rho)-\rho_c)\\ \le&(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1} - (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1} - (1+\rho_c)(\gamma_n(\rho)-\rho_c)\\ =&(1+\rho)(\gamma_{n+1}(\rho)-\rho_c) - (1+\rho_c)(\gamma_n(\rho)-\rho_c)\\ =&(1+\rho_c)\left[\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1) - (1+\rho) - \gamma_n(\rho) + \rho_c\right]\\ =&(1+\rho_c)(\rho_c-\rho) \le 0,\end{aligned}$$

where the first inequality follows from $\partial_{y_{n+1}}\tilde{G}^M_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} \ge \partial_{y_{n+1}}\tilde{G}^M_{n+1}\,|_{y_{n+1}=\tilde{M}_{n+1}} = 0$ (due to convexity), and the last equation follows from the fact that $\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1) = \gamma_n(\rho)+1$. At the kink point $y_n = \tilde{x}_n + \tilde{w}_n$, we have

$$\begin{aligned}&\partial_{y_n}\tilde{G}_n\,|_{y_n\nearrow(\tilde{x}_n+\tilde{w}_n)} - \partial_{y_n}\tilde{G}_n\,|_{y_n\searrow(\tilde{x}_n+\tilde{w}_n)}\\ =&\partial_{y_n}\tilde{G}^M_n\,|_{y_n\nearrow(\tilde{x}_n+\tilde{w}_n)} - \partial_{y_n}\tilde{G}^L_n\,|_{y_n\searrow(\tilde{x}_n+\tilde{w}_n)}\\ =&\partial_{y_n}\tilde{H}^M_n + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n}\tilde{H}^L_n + (1+\rho_c)\rho_s\right]\\ =&(1+\rho)\partial_{y_{n+1}}\tilde{G}^U_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1} - (1+\rho_c)(\rho_s-\gamma_n(\rho))\\ \le&(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1} - (1+\rho_c)(\rho_s-\gamma_n(\rho))\\ \le&-(1+\rho_c)(\rho_s-\gamma_n(\rho)) \le 0\end{aligned}$$

where the first inequality follows from $\partial_{y_{n+1}}\tilde{G}^U_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} \le \partial_{y_{n+1}}\tilde{G}^U_{n+1}\,|_{y_{n+1}=U} = 0$ (due to convexity), and the second inequality follows from $\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} \le \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}$ in the induction assumption of claim (ii). Therefore, $\tilde{G}_n$ is convex in y_n at the two kink points.

Second, with respect to $\tilde{x}_n$, we have

$$\partial_{\tilde{x}_n}\tilde{H}^U_n = \partial_{\tilde{x}_n}\tilde{H}^L_n = (1+\rho)\partial_{y_{n+1}}\tilde{G}^i_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1}, \quad i\in\{U,M\},$$

$$\partial_{\tilde{x}_n}\tilde{H}_n^M = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^i, \quad i \in \{U, M\}.$$

By using the induction assumption of claim (ii) and the fact that $\partial_{y_{n+1}}\tilde{G}_{n+1}^M |_{y_{n+1}=\tilde{x}_{n+1}} \geq 0$ and $\partial_{y_{n+1}}\tilde{G}_{n+1}^U |_{y_{n+1}=\tilde{x}_{n+1}} \leq 0$, it can be verified that $\tilde{G}_n$ is convex in $\tilde{x}_n$ at the two kink points.

Finally, with respect to $\tilde{w}_n$, we have

$$\partial_{\tilde{w}_n}\tilde{H}_n^U = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^i, \quad i \in \{U, M\},$$
$$\partial_{\tilde{w}_n}\tilde{H}_n^L = (1+\rho)\partial_{y_{n+1}}\tilde{G}_{n+1}^M |_{y_{n+1}=\tilde{x}_{n+1}} + (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^i, \quad i \in \{U, M\}.$$

Again, by using the induction assumption of claim (ii) and the fact that $\partial_{y_{n+1}}\tilde{G}_{n+1}^M |_{y_{n+1}=\tilde{x}_{n+1}} \geq 0$ and $\partial_{y_{n+1}}\tilde{G}_{n+1}^U |_{y_{n+1}=\tilde{x}_{n+1}} \leq 0$, it can be verified that $\tilde{G}_n$ is convex in $\tilde{w}_n$ at the two kink points. Therefore, we have verified that $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$.

(5) For any ξ_n such that $\tilde{x}_{n+1} \geq U$: First, with respect to y_n, we have

$$\partial_{y_n}\tilde{H}_n^U = \partial_{y_n}\tilde{H}_n^L = 0, \quad \partial_{y_n}\tilde{H}_n^M = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U.$$

At the kink point $y_n = \tilde{x}_n$, we have

$$\begin{aligned}
&\partial_{y_n}\tilde{G}_n |_{y_n \nearrow \tilde{x}_n} - \partial_{y_n}\tilde{G}_n |_{y_n \searrow \tilde{x}_n} \\
=&\partial_{y_n}\tilde{G}_n^U |_{y_n \nearrow \tilde{x}_n} - \partial_{y_n}\tilde{G}_n^M |_{y_n \searrow \tilde{x}_n} \\
=&\partial_{y_n}\tilde{H}_n^U + (1+\rho_c)\rho_c - \left[\partial_{y_n}\tilde{H}_n^M + (1+\rho_c)\gamma_n(\rho)\right] \\
=&(1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U - (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U - (1+\rho_c)(\gamma_n(\rho) - \rho_c) \\
\leq&(1+\rho)(\gamma_{n+1}(\rho) - \rho_c) - (1+\rho_c)(\gamma_n(\rho) - \rho_c) \\
=&(1+\rho_c)\left[\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1) - (1+\rho) - \gamma_n(\rho) + \rho_c\right] \\
=&(1+\rho_c)(\rho_c - \rho) \leq 0,
\end{aligned}$$

where the first inequality follows from $\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U - \partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U \leq \gamma_{n+1}(\rho) - \rho_c$ in the induction assumption of claim (ii), and the last equation follows from the fact that $\frac{1+\rho}{1+\rho_c}(\gamma_{n+1}(\rho)+1) = \gamma_n(\rho)+1$. At the kink point $y_n = \tilde{x}_n + \tilde{w}_n$, we have

$$\begin{aligned}
&\partial_{y_n}\tilde{G}_n |_{y_n \nearrow (\tilde{x}_n+\tilde{w}_n)} - \partial_{y_n}\tilde{G}_n |_{y_n \searrow (\tilde{x}_n+\tilde{w}_n)} \\
=&\partial_{y_n}\tilde{G}_n^M |_{y_n \nearrow (\tilde{x}_n+\tilde{w}_n)} - \partial_{y_n}\tilde{G}_n^L |_{y_n \searrow (\tilde{x}_n+\tilde{w}_n)} \\
=&\partial_{y_n}\tilde{H}_n^M + (1+\rho_c)\gamma_n(\rho) - \left[\partial_{y_n}\tilde{H}_n^L + (1+\rho_c)\rho_s\right] \\
=&(1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U - (1+\rho_c)(\rho_s - \gamma_n(\rho)) \\
\leq& -(1+\rho_c)(\rho_s - \gamma_n(\rho)) \leq 0,
\end{aligned}$$

where the first inequality follows from $\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} \le \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}$ in the induction assumption of claim (ii). Therefore, $\tilde{G}_n$ is convex in y_n at the two kink points.

Second, with respect to $\tilde{x}_n$, we have

$$\partial_{\tilde{x}_n}\tilde{H}^U_n = \partial_{\tilde{x}_n}\tilde{H}^L_n = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1}, \quad \partial_{\tilde{x}_n}\tilde{H}^M_n = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}.$$

By using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{x}_n$ at the two kink points.

Finally, with respect to $\tilde{w}_n$, we have

$$\partial_{\tilde{w}_n}\tilde{H}^U_n = \partial_{\tilde{w}_n}\tilde{H}^M_n = (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}, \quad \partial_{\tilde{w}_n}\tilde{H}^L_n = (1+\rho)\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1}.$$

Again, by using the induction assumption of claim (ii), it can be verified that $\tilde{G}_n$ is convex in $\tilde{w}_n$ at the two kink points. Therefore, we have verified that $\tilde{G}_n(y_n, \tilde{x}_n, \tilde{w}_n)$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$.

We have shown that given ξ_n, $\tilde{G}_n$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$. Taking expectation over ξ_n yields that $\tilde{G}_n$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$ at the two kink points $y_n = \tilde{x}_n$ and $y_n = \tilde{x}_n + \tilde{w}_n$. Together with the convexity of $\tilde{G}^i_n$ for $i \in \{U, M, L\}$ in three subproblems, it follows that $\tilde{G}_n$ is convex in y_n, $\tilde{x}_n$, and $\tilde{w}_n$, which verifies that claim (i) holds for period n.

Now, based on the first-order derivative of $\tilde{H}^i_n$ for $i \in \{U, M, L\}$, in the five cases above, we verify that claim (ii) holds for period n. We first observe from the five cases that $\partial_{\tilde{x}_n}\tilde{H}^i_n = \partial_{\tilde{w}_n}\tilde{H}^i_n$ for $i \in \{M, L\}$. This implies that

$$\partial_{\tilde{x}_n}\tilde{G}^M_n - \partial_{\tilde{w}_n}\tilde{G}^M_n = (\delta\partial_{\tilde{x}_n}\tilde{H}^M_n - \gamma_n(\rho) + \rho_c) - \delta\partial_{\tilde{w}_n}\tilde{H}^M_n = \rho_c - \gamma_n(\rho).$$

It can be similarly verified that $\partial_{\tilde{x}_n}\tilde{G}^L_n - \partial_{\tilde{w}_n}\tilde{G}^L_n = \rho_c - \gamma_n(\rho)$. Moreover, we find that $\partial_{\tilde{x}_n}\tilde{H}^U_n - \partial_{\tilde{w}_n}\tilde{H}^U_n = (1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}), i \in \{U, M, L\}$, in the cases of (1)-(3) and (5), and $\partial_{\tilde{x}_n}\tilde{H}^U_n - \partial_{\tilde{w}_n}\tilde{H}^U_n \ge (1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1})$ in the case of (4). Thus, we have: for $i \in \{U, M, L\}$,

$$\begin{aligned}\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n = \delta\partial_{\tilde{x}_n}\tilde{H}^U_n - \delta\partial_{\tilde{w}_n}\tilde{H}^U_n \ge& \delta(1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}) \\ \ge& \delta(1+\rho)(\rho_c - \gamma_{n+1}(\rho)) = (1+\rho) - \delta(1+\rho)(1+\gamma_{n+1}(\rho)) \\ =& \rho - \gamma_n(\rho) \ge \rho_c - \gamma_n(\rho),\end{aligned}$$

where the second inequality follows from $\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1} \ge \rho_c - \gamma_{n+1}(\rho), i \in \{U, M, L\}$ in the induction assumption of claim (ii). Furthermore, in the cases of (4), we also observe that $\partial_{\tilde{x}_n}\tilde{H}^U_n - \partial_{\tilde{w}_n}\tilde{H}^U_n \le (1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1})$. Thus, we have

$$\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n = \delta\partial_{\tilde{x}_n}\tilde{H}^U_n - \delta\partial_{\tilde{w}_n}\tilde{H}^U_n \le \delta(1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}) \le 0,$$

where the second inequality follows from $\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1} \le 0, i \in \{U, M, L\}$ in the induction assumption of claim (ii). It remains to check that $\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n$ only depends on the cash balance $\tilde{x}_n$. As we have discussed above, in the cases of (1)-(3), we have $\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n = \delta(\partial_{\tilde{x}_n}\tilde{H}^U_n - \partial_{\tilde{w}_n}\tilde{H}^U_n) = \delta(1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}), i \in \{M, L\}$, which is independent of both $\tilde{x}_n$ and $\tilde{w}_n$ from the induction assumption of claim (ii). In the case of (4), we have

$$\begin{aligned}\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n =& \delta(\partial_{\tilde{x}_n}\tilde{H}^U_n - \partial_{\tilde{w}_n}\tilde{H}^U_n)\\ =& \delta(1+\rho)\left[\partial_{y_{n+1}}\tilde{G}^U_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + \partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}\right],\end{aligned}$$

with $\tilde{x}_{n+1} = (1+\rho)\tilde{x}_n - \xi_n$ corresponding to the "U" case (i.e., $y_n \le \tilde{x}_n$). Since $\partial_{y_{n+1}}\tilde{H}^U_{n+1} = 0$, then the first term only depends on $\tilde{x}_{n+1}$, through which it only depends on $\tilde{x}_n$ from its corresponding state transitions. Also, $\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}$ depends on $\tilde{x}_{n+1}$ in the induction assumption of claim (ii), through which it only depends on $\tilde{x}_n$ from its corresponding state transitions. Therefore, $\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n$ only depends on $\tilde{x}_n$ in the case of (4). It can be similarly verified that $\partial_{\tilde{x}_n}\tilde{G}^U_n - \partial_{\tilde{w}_n}\tilde{G}^U_n$ only depends on $\tilde{x}_n$ in the case of (5). Altogether, we have verified claim (ii) for period n.

We next verify the claims (iii) and (iv) for period n. The proof is similar to the proof of Proposition 2. In the following, we only focus on their difference in the optimal cash policy, that is, all three optimal cash levels in the approximate system are independent of the cash balances $\tilde{x}_n$ and $\tilde{w}_n$. Similar to (13), we define the optimal cash levels $\tilde{y}^U_n(\tilde{x}_n, \tilde{w}_n)$, $\tilde{y}^M_n(\tilde{x}_n, \tilde{w}_n)$ and $\tilde{y}^L_n(\tilde{x}_n, \tilde{w}_n)$ as the unconstrained optimal solutions to the three subproblems:

$$\begin{aligned}\tilde{y}^U_n(\tilde{x}_n, \tilde{w}_n) &= \min\{y_n \mid \partial_{y_n}\tilde{G}^U_n \ge 0\} = \min\{y_n \mid \rho_c + \rho_d F(y_n) - \rho_d + \delta\partial_{y_n}\tilde{H}^U_n \ge 0\},\\ \tilde{y}^M_n(\tilde{x}_n, \tilde{w}_n) &= \min\{y_n \mid \partial_{y_n}\tilde{G}^M_n \ge 0\} = \min\{y_n \mid \gamma_n(\rho) + \rho_d F(y_n) - \rho_d + \delta\partial_{y_n}\tilde{H}^M_n \ge 0\},\\ \tilde{y}^L_n(\tilde{x}_n, \tilde{w}_n) &= \min\{y_n \mid \partial_{y_n}\tilde{G}^L_n \ge 0\} = \min\{y_n \mid \rho_s + \rho_d F(y_n) - \rho_d + \delta\partial_{y_n}\tilde{H}^L_n \ge 0\}.\end{aligned} \tag{A.20}$$

From the first-order derivative of $\tilde{H}^i_n$ for $i \in \{U, M, L\}$, in the five cases above, we have $\partial_{y_n}\tilde{H}^U_n = \partial_{y_n}\tilde{H}^L_n = 0$. This implies that $\tilde{y}^U_n(\tilde{x}_n, \tilde{w}_n) = U = F^{-1}(\frac{\rho_d - \rho_c}{\rho_d})$ and $\tilde{y}^L_n(\tilde{x}_n, \tilde{w}_n) = L = F^{-1}(\frac{\rho_d - \rho_s}{\rho_d})$.

Furthermore, from $\partial_{y_n}\tilde{H}^M_n$ in the five cases above, we have $\partial_{y_n}\tilde{H}^M_n = (1+\rho)(\partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - (1+\rho)\partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}) = (1+\rho)(\rho_c - \gamma_{n+1}(\rho)), i \in \{M, L\}$, in the cases of (1)-(3), which is independent of both $\tilde{x}_n$ and $\tilde{w}_n$ from the induction assumption of claim (ii). Combining all cases, we have

$$\begin{aligned}\partial_{y_n}\tilde{H}^M_n =& \int_{y_n - \tilde{M}_{n+1}/(1+\rho)}^{+\infty} (1+\rho)(\rho_c - \gamma_{n+1}(\rho)) f(\xi) d\xi\\ &+ \int_{y_n - U/(1+\rho)}^{y_n - \tilde{M}_{n+1}/(1+\rho)} (1+\rho)\left[\partial_{y_{n+1}}\tilde{G}^i_{n+1}\,|_{y_{n+1}=\tilde{x}_{n+1}} + \partial_{\tilde{x}_{n+1}}\tilde{G}^i_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^i_{n+1}\right] f(\xi) d\xi\\ &+ \int_{-\infty}^{y_n - U/(1+\rho)} (1+\rho)\left[\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1} - \partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}\right] f(\xi) d\xi.\end{aligned} \tag{A.21}$$

Note that, the term $(\partial_{y_{n+1}}\tilde{G}^U_{n+1}\mid_{y_{n+1}=\tilde{x}_{n+1}}+\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1}-\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1})$ only depends on $\tilde{x}_{n+1}$ (see the proof of claim (ii) above), which does not depend on $\tilde{x}_n$ and $\tilde{w}_n$ from its corresponding state transition $\tilde{x}_{n+1}=(1+\rho)y_n-\xi_n$. It can be similarly verified that $(\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1}-\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1})$ is also independent of both $\tilde{x}_n$ and $\tilde{w}_n$. Therefore, $\partial_{y_n}\tilde{H}^M_n$ and thus $\tilde{y}^M_n(\tilde{x}_n,\tilde{w}_n)$ is independent of both $\tilde{x}_n$ and $\tilde{w}_n$. We then denote it as $\tilde{M}_n$. Therefore, all three optimal cash levels are independent of the cash balances $\tilde{x}_n$ and $\tilde{w}_n$. The remaining proof of claims (iii) and (iv) is the same as the proof of Proposition 2.

Optimal middle cash level $\tilde{M}_n$. We prove $\tilde{M}_{n+1}\geq\tilde{M}_n$ for any $1\leq n<N$ in the optimal cash policy. We first verify it for the last period. We then assume it holds for period $n+1$ and prove that it is also true for period n, completing the backward induction. Based on the definition of $\tilde{M}_n$ in A.20, the sufficient condition of $\tilde{M}_{n+1}\geq\tilde{M}_n$ can be written as $\partial_{y_n}\tilde{H}^M_n-\partial_{y_{n+1}}\tilde{H}^M_{n+1}\geq(1+\rho_c)(\gamma_{n+1}(\rho)-\gamma_n(\rho))$. In the last period N, we have

$$
\begin{aligned}
&\partial_{y_{N-1}}\tilde{H}^M_{N-1}-\partial_{y_N}\tilde{H}^M_N=\partial_{y_{N-1}}\tilde{H}^M_{N-1}\\
=&\int_{y_{N-1}-\tilde{M}_N/(1+\rho)}^{+\infty}(1+\rho)(\rho_c-\gamma_N(\rho))f(\xi)d\xi\\
&+\int_{y_{N-1}-U/(1+\rho)}^{y_{N-1}-\tilde{M}_N/(1+\rho)}(1+\rho)\left[\partial_{y_N}\tilde{G}^M_N\mid_{y_N=\tilde{x}_N}+\partial_{\tilde{x}_N}\tilde{G}^M_N-\partial_{\tilde{w}_N}\tilde{G}^M_N\right]f(\xi)d\xi\\
&+\int_{-\infty}^{y_{N-1}-U/(1+\rho)}(1+\rho)\left[\partial_{\tilde{x}_N}\tilde{G}^U_N-\partial_{\tilde{w}_N}\tilde{G}^U_N\right]f(\xi)d\xi\\
\geq&\int_{y_{N-1}-\tilde{M}_N/(1+\rho)}^{+\infty}(1+\rho)(\rho_c-\gamma_N(\rho))f(\xi)d\xi+\int_{y_{N-1}-U/(1+\rho)}^{y_{N-1}-\tilde{M}_N/(1+\rho)}(1+\rho)(\rho_c-\gamma_N(\rho))f(\xi)d\xi\\
&+\int_{-\infty}^{y_{N-1}-U/(1+\rho)}(1+\rho)(\rho_c-\gamma_N(\rho))f(\xi)d\xi\\
=&(1+\rho)(\rho_c-\gamma_N(\rho))=(1+\rho_c)(\gamma_N(\rho)-\gamma_{N-1}(\rho)).
\end{aligned}
$$

It follows that $\tilde{M}_N\geq\tilde{M}_{N-1}$. Suppose that $\tilde{M}_{n+2}\geq\tilde{M}_{n+1}$ for any $1\leq n<N-1$ in the optimal cash policy. We then need to check $\tilde{M}_{n+1}\geq\tilde{M}_n$. Given the same cash level $y)n=y_{n+1}$,

$$
\begin{aligned}
\frac{1}{1+\rho}\partial_{y_n}\tilde{H}^M_n=&\int_{y_n-\tilde{M}_{n+1}/(1+\rho)}^{+\infty}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi\\
&+\int_{y_n-U/(1+\rho)}^{y_n-\tilde{M}_{n+1}/(1+\rho)}\left[\partial_{y_{n+1}}\tilde{G}^M_{n+1}\mid_{y_{n+1}=\tilde{x}_{n+1}}+\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1}-\partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1}\right]f(\xi)d\xi\\
&+\int_{-\infty}^{y_n-U/(1+\rho)}\left[\partial_{\tilde{x}_{n+1}}\tilde{G}^U_{n+1}-\partial_{\tilde{w}_{n+1}}\tilde{G}^U_{n+1}\right]f(\xi)d\xi\\
=&\int_{y_n-\tilde{M}_{n+2}/(1+\rho)}^{+\infty}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi+\int_{y_n-\tilde{M}_{n+1}/(1+\rho)}^{y_n-\tilde{M}_{n+2}/(1+\rho)}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi\\
&+\int_{y_n-U/(1+\rho)}^{y_n-\tilde{M}_{n+2}/(1+\rho)}\left[\partial_{y_{n+1}}\tilde{G}^M_{n+1}\mid_{y_{n+1}=\tilde{x}_{n+1}}+\partial_{\tilde{x}_{n+1}}\tilde{G}^M_{n+1}-\partial_{\tilde{w}_{n+1}}\tilde{G}^M_{n+1}\right]f(\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{y_n-\tilde{M}_{n+2}/(1+\rho)}^{y_n-\tilde{M}_{n+1}/(1+\rho)}\left[\partial_{y_{n+1}}\tilde{G}_{n+1}^M\mid_{y_{n+1}=\tilde{x}_{n+1}}+\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M\right]f(\xi)d\xi\\
&+\int_{-\infty}^{y_n-U/(1+\rho)}\left[\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U\right]f(\xi)d\xi\\
\ge&\int_{y_n-\tilde{M}_{n+2}/(1+\rho)}^{+\infty}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi+\int_{y_n-\tilde{M}_{n+1}/(1+\rho)}^{y_n-\tilde{M}_{n+2}/(1+\rho)}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi\\
&+\int_{y_n-U/(1+\rho)}^{y_n-\tilde{M}_{n+2}/(1+\rho)}\left[\partial_{y_{n+1}}\tilde{G}_{n+1}^M\mid_{y_{n+1}=\tilde{x}_{n+1}}+\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M\right]f(\xi)d\xi\\
&+\int_{y_n-\tilde{M}_{n+2}/(1+\rho)}^{y_n-\tilde{M}_{n+1}/(1+\rho)}(\rho_c-\gamma_{n+1}(\rho))f(\xi)d\xi+\int_{-\infty}^{y_n-U/(1+\rho)}\left[\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^U-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^U\right]f(\xi)d\xi\\
=&\frac{1}{1+\rho}\partial_{y_{n+1}}\tilde{H}_{n+1}^M+\int_{y_n-\tilde{M}_{n+2}/(1+\rho)}^{+\infty}(\gamma_{n+2}(\rho)-\gamma_{n+1}(\rho))f(\xi)d\xi\\
\ge&\frac{1}{1+\rho}\partial_{y_{n+1}}\tilde{H}_{n+1}^M+\gamma_{n+2}(\rho)-\gamma_{n+1}(\rho),
\end{aligned}
$$

where the first inequality follows from $\partial_{y_{n+1}}\tilde{G}_{n+1}^M\mid_{y_{n+1}=\tilde{x}_{n+1}}\ge\partial_{y_{n+1}}\tilde{G}_{n+1}^M\mid_{y_{n+1}=\tilde{M}_{n+1}}=0$ in the case of $\tilde{M}_{n+1}\le\tilde{x}_{n+1}<U$ and $\partial_{\tilde{x}_{n+1}}\tilde{G}_{n+1}^M-\partial_{\tilde{w}_{n+1}}\tilde{G}_{n+1}^M=\rho_c-\gamma_{n+1}(\rho_c)$ in the claim (ii), and the last inequality follows from the fact that $\gamma_{n+2}(\rho_c)<\gamma_{n+1}(\rho_c)$. Rearranging the inequality above yields

$$
\begin{aligned}
\partial_{y_n}\tilde{H}_n^M-\partial_{y_{n+1}}\tilde{H}_{n+1}^M&\ge(1+\rho)(\gamma_{n+2}(\rho)-\gamma_{n+1}(\rho_))\\
&=(1+\rho_c)\left[\frac{1+\rho}{1+\rho_c}(1+\gamma_{n+2}(\rho))-\frac{1+\rho}{1+\rho_c}(1+\gamma_{n+1}(\rho))\right]\\
&=(1+\rho_c)(\gamma_{n+1}(\rho)-\gamma_n(\rho)).
\end{aligned}
$$

Therefore, for any $1\le n\le N$, we have $\tilde{M}_{n+1}\ge\tilde{M}_n$. This completes the proof. □

Proof of Proposition 7

For each period n, the cash level decision is independent of the future periods and only minimizes the cash flow cost in the current period. This is similar to our original problem $G_n(y_n,x_n,w_n)$ when $n=N$, since $V_{N+1}(\cdot,\cdot)=0$. As we have shown in the proof of Proposition 2, the original objective function in the last period $G_N(y_N,x_N,w_N)$ is convex in y_N,x_N and w_N. This implies that $G_n^m(y_n,x_n,w_n)$ is also convex in y_n,x_n and w_n. Correspondingly, the three optimal cash levels and critical levels in this myopic problem are equivalent, which are denoted as $L=F^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)$, $M_n^m=F^{-1}\left(\frac{\rho_d-\gamma_n(\rho)}{\rho_d}\right)$, and $U=F^{-1}\left(\frac{\rho_d-\rho_c}{\rho_d}\right)$. These three cash levels satisfy $L\le M_n^m\le U$ as proved in the Proposition 2. Therefore, the myopic cash policy can be written as

$$y_n^m = \begin{cases} L = F^{-1}\left(\frac{\rho_d - \rho_s}{\rho_d}\right) & \text{if } x_n + w_n < L, \\ x_n + w_n & \text{if } L \le x_n + w_n < M_n, \\ M_n = F^{-1}\left(\frac{\rho_d - \gamma_n(\rho)}{\rho_d}\right) & \text{if } x_n < M_n \le x_n + w_n, \\ x_n & \text{if } M_n \le x_n < U, \\ U = F^{-1}\left(\frac{\rho_d - \rho_c}{\rho_d}\right) & \text{if } x_n \ge U. \end{cases}$$

We finally prove $M_n^m = F^{-1}(\frac{\rho_d - \gamma_n(\rho)}{\rho_d}) \le \tilde{M}_n$ for $1 \le n \le N$. From the definition of $\tilde{M}_n$ in (A.20), the convexity of $\tilde{G}_n^M$ in y_n implies that $\partial_{y_n}\tilde{G}_n^M$ is increasing in y_n. From equation (A.21) of $\partial_{y_n}\tilde{H}_n^M$, it can be easily verified that $\partial_{y_n}\tilde{H}_n^M \le 0$ due to the claim (ii) and $\partial_{y_{n+1}}\tilde{G}_{n+1}^U |_{y_{n+1}=\tilde{x}_{n+1}} \le \partial_{y_{n+1}}\tilde{G}_{n+1}^U |_{y_{n+1}=U} = 0$ in the case of (4) of $\tilde{V}_{n+1}$ above (see the proof of Proposition 6). Thus, we have $\partial_{y_n}\tilde{G}_n^M \le \gamma_n(\rho) + \rho_d F(y_n) - \rho_d$. Due to an increase of $\partial_{y_n}\tilde{G}_n^M$ in y_n, we have $\tilde{M}_n \ge M_n^m = F^{-1}(\frac{\rho_d - \gamma_n(\rho)}{\rho_d})$ for $1 \le n \le N$. This completes the proof. □

B. Decoupling Cash Balance Model

We analyze the payables finance problem under the decoupling cash balance assumption of the classic cash flow literature. We first derive the decoupling cash balance model. Based on which we characterize its corresponding optimal cash policy.

According to the decoupling assumption, the interest gains and costs are assumed to be separated from the cash balance itself. Essentially, the decoupling cash balance model features two separate accounts, the cash pool and the interest account, for cash management. The cash balance evolves in the cash pool, and the interest gains and costs incurred from the cash pool are in the interest account so as to evaluate the supplier's cash management performance. Thus, the cash pool evolves without being affected by the associated interest gains and costs.

At the beginning of period n, the supplier has an initial cash balance $\hat{x}_n$ in the cash pool and $\hat{u}_n$ in the separate interest account. Also let $\hat{w}_n$ be the net cash amount (after discounting) available to the supplier to draw from payables finance. The initial balances in the cash pool are still the same as in the integrated cash balance model, that is, $\hat{x}_1 = X$ and $\hat{w}_1 = W/(1+\rho)^N$. Besides, the initial balance in the separate interest account is $\hat{u}_1 = 0$. Specifically, in the cash pool the cash balance in period $n+1$ (with $1 \le n \le N$) can be written as

$$\hat{x}_{n+1} = y_n - \xi_n + (\hat{x}_n - y_n)^+ - (y_n - \hat{x}_n - \hat{w}_n)^+, \tag{A.22}$$

where $(x)^+ = \max\{x, 0\}$. As discussed in §3, the third term in the above expression is the amount from investment (if any), and the last term is the amount borrowed from additional short-term loans (if any) after the supplier withdraws all available amount from payables finance. We note that the interest rates of both terms are not accounted for the cash balance in the cash pool under the decoupling assumption. Moreover, the period-to-period cash balance transition (2) for the withdrawable payables finance amount stays the same (determined by its account establishment), but grows with the above decoupling cash balance in the cash pool. That is, for $1 \le n \le N$,

$$\hat{w}_{n+1} = (1+\rho)\left[\hat{w}_n - (y_n - \hat{x}_n)^+\right]^+. \tag{A.23}$$

In addition, the cash balance in the separate interest account in period $n+1$ (with $1 \le n \le N$) can be written as

$$\hat{u}_{n+1} = (1+\rho_c)\hat{u}_n - \rho_d(\xi_n - y_n)^+ + \rho_c(\hat{x}_n - y_n)^+ - \rho_s(y_n - \hat{x}_n - \hat{w}_n)^+ + (\rho - \rho_c)\left[\hat{w}_n - (y_n - \hat{x}_n)^+\right]^+.$$

The first term captures the risk-free interest return of owning cash balance $\hat{u}_n$ in period n, the second term denotes the interest cost if the ending balance is negative in the period, the third term is the risk-free interest gain from investments, the fourth term represents the interest cost of using additional short-term loans, and the last term denotes the extra earning interest of keeping the payables finance amount unused.

As discussed above, the separate interest account is intended for evaluating the supplier's cash management performance in the decoupling cash balance model. Therefore, different from the integrated cash balance model, the objective for the supplier is to maximize the discounted total cash balance in the interest account, denoted as $\hat{\Pi}(X,W)$, at the beginning of the terminal period $N+1$, which can be written as

$$\hat{\Pi}(X,W) = \max_{\{y_1,\dots,y_N\}} \delta^N E\left[\hat{u}_{N+1}\right], \tag{A.24}$$

where $\{y_1,...,y_N\}$ is the unconstrained cash level policy for each period before the payment due date N. With some term substitution and rearrangement, we can transform problem (A.24) into a cost minimization problem as follows (see the proof of Proposition B.1 in Appendix A):

Proposition B.1. *The following holds:*

$$\hat{\Pi}(X,W) = \delta^N W - \frac{W}{(1+\rho)^N} - \delta\hat{V}_1(\hat{x}_1, \hat{w}_1), \tag{A.25}$$

where $\hat{V}_1(\hat{x}_1, \hat{w}_1)$ is determined by the dynamic program: for $1 \le n \le N$,

$$\begin{aligned}\hat{V}_n(\hat{x}_n, \hat{w}_n) = \min_{y_n} \Big\{ &\rho_c y_n + \rho_d E\left[(\xi_n - y_n)^+\right] + (\gamma_n(\rho) - \rho_c)\min\left\{(y_n - \hat{x}_n)^+, \hat{w}_n\right\} \\ &+ (\rho_s - \rho_c)(y_n - \hat{x}_n - \hat{w}_n)^+ + \delta E\left[\hat{V}_{n+1}(\hat{x}_{n+1}, \hat{w}_{n+1})\right]\Big\}, \end{aligned} \tag{A.26}$$

$$\hat{V}_{N+1}(\cdot,\cdot) = 0,$$

with $\gamma_n(\rho) = \delta^{N-n}(1+\rho)^{N-n+1} - 1$, $\hat{x}_1 = X$, $\hat{w}_1 = (1+\rho)^{-N}W$, and $\hat{x}_{n+1}$ and $\hat{w}_{n+1}$ given in (A.22) to (A.23).

Proof. Similar to the proof of Proposition 1, we write the discounted cash balance in the interest account according to the following sum of discounted incremental cash balances of each period:

$$\delta^N E[\hat{u}_{N+1}] = \hat{u}_1 + \sum_{n=1}^{N} \delta^{n-1} E\left[\delta \hat{u}_{n+1} - \hat{u}_n\right] = \sum_{n=1}^{N} \delta^{n-1} E\left[\delta \hat{u}_{n+1} - \hat{u}_n\right].$$

Substituting the expressions of $\hat{u}_{n+1}$ into the right-hand side yields

$$\begin{aligned}&\sum_{n=1}^{N} \delta^{n-1} E\left[\delta \hat{u}_{n+1} - \hat{u}_n\right] \\ =&\delta \sum_{n=1}^{N} \delta^{n-1} E\left[-\rho_d(\xi_n - y_n)^+ + \rho_c(\hat{x}_n - y_n)^+ - \rho_s(y_n - \hat{x}_n - \hat{w}_n)^+ + (\rho - \rho_c)\left[\hat{w}_n - (y_n - \hat{x}_n)^+\right]^+\right].\end{aligned}$$

Note that, $[\hat{w}_n - (y_n - \hat{x}_n)^+]^+ = \hat{w}_n - (y_n - \hat{x}_n)^+ + (y_n - \hat{x}_n - \hat{w}_n)^+$.

As we have shown in the proof of Proposition 1,

$$\begin{aligned}&(\rho - \rho_c)\hat{w}_n \\ =&(\rho - \rho_c)(1+\rho)^{n-1}\hat{w}_1 - (\rho - \rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i - \hat{x}_i)^+ + (\rho - \rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i - \hat{x}_i - \hat{w}_i)^+.\end{aligned}$$

Substituting the above expression back, we have

$$\begin{aligned}&\sum_{n=1}^{N} \delta^{n-1} E\left[\delta \hat{u}_{n+1} - \hat{u}_n\right] \\ =&\delta \sum_{n=1}^{N} \delta^{n-1} \left\{-\rho_d E(\xi_n - y_n)^+ + \rho_c(\hat{x}_n - y_n)^+ - \rho_s(y_n - \hat{x}_n - \hat{w}_n)^+ + (\rho - \rho_c)\hat{w}_n \right.\\ &\left. -(\rho - \rho_c)[(y_n - \hat{x}_n)^+ - (y_n - \hat{x}_n - \hat{w}_n)^+]\right\} \\ =&\delta \sum_{n=1}^{N} \delta^{n-1} \left\{-\rho_d E(\xi_n - y_n)^+ + \rho_c(\hat{x}_n - y_n)^+ - \rho_s(y_n - \hat{x}_n - \hat{w}_n)^+ + (\rho - \rho_c)(1+\rho)^{n-1}\hat{w}_1 \right.\\ &-(\rho - \rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i - \hat{x}_i)^+ + (\rho - \rho_c)\sum_{i=1}^{n-1}(1+\rho)^{n-i}(y_i - \hat{x}_i - \hat{w}_i)^+ \\ &\left. -(\rho - \rho_c)[(y_n - \hat{x}_n)^+ - (y_n - \hat{x}_n - \hat{w}_n)^+]\right\} \\ =&\left[\delta^N - \frac{1}{(1+\rho)^N}\right] W \\ &- \delta \sum_{n=1}^{N} \delta^{n-1} \left\{\rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)\min\left\{(y_n - \hat{x}_n)^+, \hat{w}_n\right\} + (\rho_s - \rho_c)(y_n - \hat{x}_n - \hat{w}_n)^+\right\}.\end{aligned}$$

Therefore, we can obtain the following equivalent cost minimization problem: for $n = 1, 2, \cdots, N$,

$$\begin{aligned}\hat{V}_n(\hat{x}_n, \hat{w}_n) = \min_{y_n} &\left\{\rho_c y_n + \rho_d E(\xi_n - y_n)^+ + (\gamma_n(\rho) - \rho_c)\min\left\{(y_n - \hat{x}_n)^+, \hat{w}_n\right\}\right.\\ &\left. +(\rho_s - \rho_c)(y_n - \hat{x}_n - \hat{w}_n)^+ + \delta E\left[\hat{V}_{n+1}(\hat{x}_{n+1}, \hat{w}_{n+1})\right]\right\},\end{aligned}$$

with $\gamma_n(\rho) = \delta^{N-n}(1+\rho)^{N-n+1} - 1$, $\hat{x}_1 = X$, $\hat{w}_1 = (1+\rho)^{-N}W$. Plugging it into the objective function $\hat{\Pi}(X, W)$, we have $\hat{\Pi}(X, W) = \delta^N W - \frac{W}{(1+\rho)^N} - \delta \hat{V}_1(\hat{x}_1, \hat{w}_1)$. □

In the objective function $\hat{\Pi}(X,W)$ in Proposition B.1, the first two terms ($\delta^N W - \frac{W}{(1+\rho)^N}$) capture the interest gain if withdrawing from payables finance until the payment due date; and the last term $\delta\hat{V}_1(\hat{x}_1,\hat{w}_1)$ is the discounted cost due to cash flow uncertainty. The difference of $\hat{\Pi}(X,W)$ from (4) in Proposition 1 is that, in the decoupling cash balance model, only the cash balance in the separate interest account is taken into consideration in the objective function. Besides, the dynamic program (A.26) in the decoupling cash balance model is the same as the dynamic program (5) in the integrated cash balance model, except for the period-to-period cash balance transitions. Based on the problem (A.26), we can characterize its optimal cash policy, which we refer to as the "decoupling cash policy" and is summarized in the following proposition.

PROPOSITION B.2. *For any $1 \le n \le N$, the optimal cash policy $\hat{y}_n^*$ for the decoupling cash balance model is given by*

$$\hat{y}_n^* = \begin{cases} L = F^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right) & \text{if } \hat{x}_n+\hat{w}_n < L, \\ \hat{x}_n+\hat{w}_n & \text{if } L \le \hat{x}_n+\hat{w}_n < \hat{M}_n, \\ \hat{y}_n^M(\hat{x}_n+\hat{w}_n) & \text{if } \hat{x}_n < \hat{M}_n \le \hat{x}_n+\hat{w}_n, \\ \hat{x}_n & \text{if } \hat{M}_n \le \hat{x}_n < U, \\ U = F^{-1}\left(\frac{\rho_d-\rho_c}{\rho_d}\right) & \text{if } \hat{x}_n \ge U, \end{cases}$$

where

$$\begin{aligned}\hat{y}_n^M(\hat{x}_n+\hat{w}_n) = &\arg\min_{y_n}\{\gamma_n(\rho)y_n - (\gamma_n(\rho)-\rho_c)\hat{x}_n + \rho_d E\left[(\xi_n-y_n)^+\right] \\ &+ \delta E\left[\hat{V}_{n+1}(y_n-\xi_n,(1+\rho)(\hat{w}_n+\hat{x}_n-y_n))\right]\},\end{aligned}$$

and $\hat{M}_n = \min\{\hat{x}_n+\hat{w}_n \mid \hat{y}_n^M(\hat{x}_n+\hat{w}_n) = \hat{x}_n+\hat{w}_n\}$, with $L < \hat{M}_n(\hat{x}_n,\hat{w}_n) < U$.

Proof. The proof is similar to Proposition 6 in the approximate system. The only difference is that the optimal middle cash level $\hat{y}_n^M(\hat{x}_n+\hat{w}_n)$ in the decoupling model depends on the cash balances. To see this, we first write the state transition equations (A.22) and (A.23) in the decoupling model as

$$\hat{x}_{n+1} = \begin{cases} \hat{x}_n - \xi_n & \text{if } y_n \le \hat{x}_n, \\ y_n - \xi_n & \text{if } \hat{x}_n < y_n \le \hat{x}_n+\hat{w}_n, \\ \hat{x}_n+\hat{w}_n-\xi_n & \text{if } y_n > \hat{x}_n+\hat{w}_n; \end{cases} \tag{A.27}$$

$$\hat{w}_{n+1} = \begin{cases} (1+\rho)\hat{w}_n & \text{if } y_n \le \hat{x}_n, \\ (1+\rho)(\hat{w}_n+\hat{x}_n-y_n) & \text{if } \hat{x}_n < y_n \le \hat{x}_n+\hat{w}_n, \\ 0 & \text{if } y_n > \hat{x}_n+\hat{w}_n. \end{cases} \tag{A.28}$$

We can find that in the cases of $y_n \le \hat{x}_n$ and $y_n > \hat{x}_n+\hat{w}_n$, both $\hat{x}_{n+1}$ and $\hat{w}_{n+1}$ are independent of y_n. As a result, their corresponding optimal cash levels are $U = F^{-1}\left(\frac{\rho_d-\rho_c}{\rho_d}\right)$ and $L = F^{-1}\left(\frac{\rho_d-\rho_s}{\rho_d}\right)$.

However, in the case of $\hat{x}_n < y_n \leq \hat{x}_n + \hat{w}_n$, the total cash balance $\hat{x}_{n+1} + \hat{w}_{n+1}$ is a function of y_n, $\hat{x}_n$, and $\hat{w}_n$. This implies that the first-order derivative of $E\left[\hat{V}_{n+1}(\hat{x}_{n+1}, \hat{w}_{n+1})\right]$ with respect to y_n in the second case depends on the cash balances $\hat{x}_n$, and $\hat{w}_n$. It follows that the optimal middle cash level $\hat{y}_n^M(\hat{x}_n + \hat{w}_n)$ depends on both $\hat{x}_n$, and $\hat{w}_n$. $\square$